Math 31CH Spring 2017 Homework 3, due 4/26/2017 in HW box in the basement of AP&M by 5 pm

1 Reading

Read Sections 5.2, 5.3. The midterm exam on April 28 will cover the material on homeworks 1-3. That includes all of the Sections of Chapter 4 we covered and Sections 5.1 and 5.2.

2 Exercises to submit on Wednesday 4/19

2.1 Exercises from the text

Section 4.10: #1, 4, 9, 12, 13, 14, 17

Section 5.1: #1, 2, 4

Section 5.2: #3

2.2 Exercise not from the text

1. The point of this problem is to give a deeper explanation of why the definition of the k-volume of a k-parallelogram in $\mathbb{R}^n$, as given in the text in Section 5.1, is reasonable. Let $\vec{v_1}, \ldots, \vec{v_k}$ be a list of linearly independent vectors in $\mathbb{R}^n$, where $k \leq n$. Let $V = \mathbb{R}\vec{v_1} + \cdots + \mathbb{R}\vec{v_k}$ be the subspace of $\mathbb{R}^n$ which is spanned by the vectors $\vec{v_i}$. Since we assumed these vectors are independent, then V is a vector space of dimension k.

(a). Show that V has an orthonormal basis. In other words, show you can find $\vec{f_1}, \ldots, \vec{f_k} \in V$ which satisfy $(\vec{f_i}, \vec{f_i}) = 1$ for all i and $(\vec{f_i}, \vec{f_j}) = 0$ for all $i \neq j$. Here, the dot product $(\, , \,)$ is the standard one in $\mathbb{R}^n$.

(Hint: Let $\vec{f_1}$ be a scalar multiple of $\vec{v_1}$ which has norm 1. Then show you can choose $\vec{f_2}$ of norm 1 which is in the span of $\vec{v_1}$ and $\vec{v_2}$ and is orthogonal to $\vec{f_1}$. Continue in this way by induction.)

(b). Show that there is a unique invertible linear transformation $S : V \to \mathbb{R}^k$ such that $S(\vec{f_i}) = \vec{e_i}$, where the $\vec{e_i}$ are the standard basis vectors in $\mathbb{R}^k$. Show that S is an *isometry* in the sense that if $\vec{w_1}, \vec{w_2} \in V$, then $(\vec{w_1}, \vec{w_2}) = (S(\vec{w_1}), S(\vec{w_2}))$, and therefore if $\vec{w} \in V$, then $||\vec{w}|| = ||S(\vec{w})||$.

(c). Part (b) shows that the map S preserves the geometry of the spaces and thus S should preserve volumes. Thus to define the k-volume of $A = P(\vec{v_1}, \ldots, \vec{v_k})$ we should be able to transfer the problem to the space $\mathbb{R}^k$ using S. Define the k-volume of A as in the text, as $\sqrt{\det(T^T T)}$. We can also define the k-volume of $S(A) \subseteq \mathbb{R}^k$ as we have already defined the k-volume of a subset of $\mathbb{R}^k$, namely as $\int_{\mathbb{R}^k} 1_{S(A)} \, |d^k \vec{x}|$. Show that these two definitions give the same number.