1. Some definitions and formulas

Theorem 1.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, and $f : \mathbb{R}^n \to \mathbb{R}$ an integrable function. Then $f \circ T$ is integrable, and*

$$\int_{\mathbb{R}^n} f(\vec{y})|d^n\vec{y}| = |\det T| \int_{\mathbb{R}^n} f(T(\vec{x}))|d^n\vec{x}|.$$

Theorem 1.2. *Let X be a compact subset of $\mathbb{R}^n$ with boundary ∂X of volume 0; let $U \subset \mathbb{R}^n$ be an open set containing X. Let $\Phi : U \to \mathbb{R}^n$ be a C^1 mapping that is injective on $(X - \partial X)$ and has Lipschitz derivative, with $[D\Phi(\vec{x})]$ invertible at every $\vec{x} \in (X - \partial X)$. Set $Y = \Phi(X)$.*

Then if $f : Y \to \mathbb{R}$ is integrable, $(f \circ \Phi)|\det[D\Phi]|$ is integrable on X and

$$\int_Y f(\vec{y})|d^n\vec{y}| = \int_X (f \circ \Phi)(\vec{x})|\det D\Phi(\vec{x})||d^n\vec{x}|.$$

Lemma 1.3. *(Spherical coordinates) If*

$$S : \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{pmatrix}$$

then $|\det[DS]| = r^2\cos\varphi$.

Lemma 1.4. *(Cylindrical coordinates) If*

$$S : \begin{pmatrix} r \\ \theta \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

then $|\det[DS]| = r$.

Definition 1.5. *Suppose that $M \subseteq \mathbb{R}^n$ is a k-dimensional manifold. Let $\gamma : U \to \mathbb{R}^n$ be a relaxed parametrization of M, where X is the set of bad points. Then*

$$\mathrm{vol}_k M = \int_{U-X} \sqrt{\det([D\gamma(\vec{u})]^T[D\gamma(\vec{u})])}|d^k\vec{u}|.$$

Definition 1.6. *let $U \subseteq \mathbb{R}^k$ be a bounded open set with $\mathrm{vol}_k \partial U = 0$. Let V be an open subset of $\mathbb{R}^n$ and let $\gamma : U \to \mathbb{R}^n$ be a C^1 mapping with $\gamma(U) \subseteq V$. Let φ be a k-form field defined on V. Then the integral of φ over $[\gamma(U)]$ is*

$$\int_{[\gamma(U)]} \varphi = \int_U \varphi(P_{\gamma(\vec{u})}(\vec{D}_1\gamma(\vec{u}), \ldots, \vec{D}_k\gamma(\vec{u})))|d^k\vec{u}|.$$

Definition 1.7. *Let $M \subseteq \mathbb{R}^n$ be a k-dimensional oriented manifold, φ a k-form field on a neighborhood of M, and $\gamma : U \to \mathbb{R}^n$ an orientation-preserving parametrization of M. Then*

$$\int_M \varphi = \int_{[\gamma(U)]} \varphi$$

as defined in the previous definition.

Theorem 1.8. *Let $U \subseteq \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^{n-k}$ be a map of class C^1 such that $[Df(\vec{x})]$ is surjective at all $\vec{x} \in M = f^{-1}(\vec{0})$. Then the map $\Omega_{\vec{x}} : B(T_{\vec{x}}(M)) \to \{-1, 1\}$ given by*

$$\Omega_{\vec{x}}(\vec{v}_1, \ldots, \vec{v}_k) = \operatorname{sgn} \det[\vec{\nabla} f_1, \ldots, \vec{\nabla} f_{n-k}, \vec{v}_1, \ldots, \vec{v}_k]$$

is an orientation of M.

I will remind you of definitions and theorems involving the specialized language of work, flux, and mass forms, and curl and div, if they are needed.