MATH 31CH SPRING 2017 MIDTERM 2

Instructions: Justify all of your answers, and show your work. You may use the result of one part of a problem in the proof of a later part, even if you do not complete the proof of the earlier part. You may quote basic theorems proved in the textbook or in class, unless the point of the problem is to reproduce the proof of such a theorem, or the problem tells you not to. Do not quote the results of homework exercises.

1 (20 pts). Let C be a smooth curve in $\mathbb{R}^2$, in other words, a 1-dimensional manifold. Suppose that for each $\vec{x} \in C$ we choose a vector $\vec{n}(\vec{x}) \in \mathbb{R}^2$ such that (i) $0 \neq \vec{n}(\vec{x})$ for all $\vec{x}$; (ii) $\vec{n}(\vec{x}) \notin T_{\vec{x}}(C)$ for all $\vec{x}$, and (iii) $\vec{n} : C \to \mathbb{R}^2$ is a continuous function.

(a) (10 pts). Prove directly from the definition of orientation that the formula $\Omega_{\vec{x}}(\vec{v}) = \operatorname{sgn}\det(\vec{n}(\vec{x}), \vec{v})$ defines an orientation of C.

(b) (5 pts). Let $C = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| -1 < x < 1, y = x^2 \right\}$. Suppose we choose the constant function $\vec{n}(\vec{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Show that part (a) applies and thus defines an orientation of C.

(c) (5 pts). With the oriented curve C given in part (b), and the 1-form field $\varphi = y\,dx + x\,dy$, calculate $\int_C \varphi$.

2 (15 pts). Given two vectors $\vec{v}_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ we define

$$\varphi(\vec{v}_1, \vec{v}_2) = (3a_1 + a_3)(2b_2 + b_4) - (3b_1 + b_3)(2a_2 + a_4).$$

(a) (5 pts). Prove directly from the definition that ϕ is a 2-form on $\mathbb{R}^4$.

(b) (5 pts). Write ϕ as an explicit linear combination of elementary 2-forms.

(c). (5 pts) Show that $\phi \wedge dx_4 \wedge dx_3 = \lambda \det$ for some scalar λ, and find λ.

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3 (20 pts). Let S be the surface given by $z = \sqrt{(x^2 + y^2)}$ for $0 < z < 1$.

(a) (10 pts). Find a parameterization of this surface and calculate its surface area.

(b) (10 pts). Let $f(x, y, z) = z - \sqrt{(x^2 + y^2)}$, so that S is the set of points where f is zero. Orient S using the gradient vector $\vec{\nabla}_f$, in words using the formula $\Omega_{\vec{x}}(\vec{v_1}, \vec{v_2}) = \operatorname{sgn}\det(\vec{\nabla}_f, \vec{v}_1, \vec{v}_2)$.

Find $\displaystyle\int_S \varphi$ for the 2-form field $\varphi = z\, dx \wedge dy$.

1. Some definitions and formulas

Definition 1.1. *Suppose that $M \subseteq \mathbb{R}^n$ is a k-dimensional manifold. Let $\gamma : U \to \mathbb{R}^n$ be a relaxed parametrization of M, where X is the set of bad points. Then*

$$\operatorname{vol}_k M = \int_{U-X} \sqrt{\det([D\gamma(\vec{u})]^T[D\gamma(\vec{u})])}|d^k\vec{u}|.$$

Definition 1.2. *let $U \subseteq \mathbb{R}^k$ be a bounded open set with $\operatorname{vol}_k \partial U = 0$. Let V be an open subset of $\mathbb{R}^n$ and let $\gamma : U \to \mathbb{R}^n$ be a C^1 mapping with $\gamma(U) \subseteq V$. Let φ be a k-form field defined on V. Then the integral of φ over $[\gamma(U)]$ is*

$$\int_{[\gamma(U)]} \varphi = \int_U \varphi(P_{\gamma(\vec{u})}(\vec{D}_1\gamma(\vec{u}), \ldots, \vec{D}_k\gamma(\vec{u})))|d^k\vec{u}|.$$

Definition 1.3. *Let $M \subseteq \mathbb{R}^n$ be a k-dimensional oriented manifold, φ a k-form field on a neighborhood of M, and $\gamma : U \to \mathbb{R}^n$ an orientation-preserving parametrization of M. Then*

$$\int_M \varphi = \int_{[\gamma(U)]} \varphi$$

as defined in the previous definition.