MATH 31CH SPRING 2017 MIDTERM 2 SOLUTIONS

1 (20 pts). Let C be a smooth curve in $\mathbb{R}^2$, in other words, a 1-dimensional manifold. Suppose that for each $\vec{x} \in C$ we choose a vector $\vec{n}(\vec{x}) \in \mathbb{R}^2$ such that (i) $0 \neq \vec{n}(\vec{x})$ for all $\vec{x}$; (ii) $\vec{n}(\vec{x}) \notin T_{\vec{x}}(C)$ for all $\vec{x}$, and (iii) $\vec{n} : C \to \mathbb{R}^2$ is a continuous function.

(a) (10 pts). Prove directly from the definition of orientation that the formula $\Omega_{\vec{x}}(\vec{v}) = \operatorname{sgn}\det(\vec{n}(\vec{x}), \vec{v})$ defines an orientation of C.

(b) (5 pts). Let $C = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| -1 < x < 1, y = x^2 \right\}$. Suppose we choose the constant function $\vec{n}(\vec{x}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Show that part (a) applies and thus defines an orientation of C.

(c) (5 pts). With the oriented curve C given in part (b), and the 1-form field $\varphi = y\,dx + x\,dy$, calculate $\int_C \varphi$.

Solution. (a). We need $\Omega(\vec{x}, \vec{v}) = \Omega_{\vec{x}}(\vec{v})$ to be a continuous function of $\vec{x} \in C$, $\vec{v} \in T_{\vec{x}}(C)$. By definition $\vec{n}(\vec{x})$ is a continous function of $\vec{x}$, so the entries of the matrix $(\vec{n}(\vec{x}), \vec{v})$ are continuous functions of $\vec{x}$ and $\vec{v}$. The determinant is a continuous function of the entries of a matrix (it is even a polynomial in the entries of the matrix), so $\det(\vec{n}(\vec{x}), \vec{v})$ is a continuous function of $\vec{x}, \vec{v}$. Finally, by assumption $\vec{n}(\vec{x})$ is not in the 1-dimensional subspace $T_{\vec{x}}(C)$ of $\mathbb{R}^2$, so necessarily $\vec{n}(\vec{x})$ and $\vec{v}$ span $\mathbb{R}^2$, and so they are also independent. Thus $\det(\vec{n}(\vec{x}), \vec{v}) \neq 0$. Since $\operatorname{sgn} : \mathbb{R} - \{0\} \to \mathbb{R}$ is continuous, we conclude that $\operatorname{sgn}\det(\vec{n}(\vec{x}), \vec{v})$ is a continuous function of $\vec{x}, \vec{v}$ as required.

We also need that for fixed $\vec{x}$, $\Omega_{\vec{x}}(\vec{v}) = \operatorname{sgn}\det(\vec{n}(\vec{x}), \vec{v})$ defines an orientation of the vector space $T_{\vec{x}}(C)$. Since $T_{\vec{x}}(C)$ is 1-dimensional, any nonzero vector in this space is a basis. If $\vec{v}$ and $\vec{v'}$ are 2 nonzero vectors in $T_{\vec{x}}(C)$, then $\vec{v'} = \lambda\vec{v}$ for some $\lambda \neq 0$, and the change of basis matrix is $P_{\vec{v'} \to \vec{v}} = [\lambda]$. We have

$$\det(\vec{n}(\vec{x}), \vec{v'}) = \det(\vec{n}(\vec{x}), \lambda\vec{v}) = \lambda \det(\vec{n}(\vec{x}), \vec{v})$$

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Thus $\Omega_{\vec{x}}(\vec{v})$ and $\Omega_{\vec{x}}(\vec{v}')$ are equal if and only if $\lambda = \det P_{\vec{v}' \to \vec{v}} > 0$, which is the definition of an orientation on a vector space.

(b). We need $\vec{n}$ to satisfy the conditions in (a). Since it is constant nonzero function, it is clearly continous and nonzero. We just need to check that $\vec{n}(\vec{x}) \notin T_{\vec{x}}(C)$. Since $\gamma : [-1,1] \to \mathbb{R}^2$ given by $\gamma(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$ is a parameterization of C, $D\gamma(t) = \begin{bmatrix} 1 \\ 2t \end{bmatrix}$ spans the tangent space to C at $\gamma(t)$. The vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is never in the space spanned by $\begin{bmatrix} 1 \\ 2t \end{bmatrix}$, as required.

(c). Consider the parameterization γ of part (b). Then for each t, $\Omega_{\gamma(t)}(D\gamma(t)) = \operatorname{sgn}\det(\vec{n}, D\gamma(t)) = \operatorname{sgn}\det \begin{bmatrix} 0 & 1 \\ 1 & 2t \end{bmatrix} = -1$. This shows that the given parameterization reverses the orientation. Rather than changing the parameterization, we simply calculate the integral then multiply by -1 at the end.

We have

$$\int_{[\gamma]} ydx + xdy = \int_{-1}^{1} (t^2)(1) + (t)(2t)dt = \int_{-1}^{1} 3t^2 dt = t^3\Big|_{-1}^{1} = 2.$$

Thus

$$\int_C ydx + xdy = -\int_{[\gamma]} ydx + xdy = -2.$$

2 (15 pts). Given two vectors $\vec{v}_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ we define

$$\varphi(\vec{v}_1, \vec{v}_2) = (3a_1 + a_3)(2b_2 + b_4) - (3b_1 + b_3)(2a_2 + a_4).$$

(a) (5 pts). Prove directly from the definition that φ is a 2-form on $\mathbb{R}^4$.

(b) (5 pts). Write φ as an explicit linear combination of elementary 2-forms.

(c). (5 pts) Show that $\varphi \wedge dx_4 \wedge dx_3 = \lambda \det$ for some scalar λ, and find λ.

Solution.

(a). (Note that since you are asked to prove (a) from the definition, you cannot prove (b) first and then claim you are done by the theorem that linear combinations of elementary 2-forms are 2-forms).

We need the function $\varphi : (\mathbb{R}^4)^2 \to \mathbb{R}$ to be multilinear and antisymmetric. If $\vec{w} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ and $\lambda, \mu \in \mathbb{R}$, then $\lambda \vec{v}_1 + \mu \vec{w} = \begin{bmatrix} \lambda a_1 + \mu c_1 \\ \lambda a_2 + \mu c_2 \\ \lambda a_3 + \mu c_3 \\ \lambda a_4 + \mu c_4 \end{bmatrix}$. Thus

$$\varphi(\lambda \vec{v}_1 + \mu \vec{w}, \vec{v}_2) = (3(\lambda a_1 + \mu c_1) + \lambda a_3 + \mu c_3)(2b_2 + b_4) - (3b_1 + b_3)(2(\lambda a_2 + \mu c_2) + \lambda a_4 + \mu c_4)$$
$$= \lambda[(3a_1 + a_3)(2b_2 + b_4) - (3b_1 + b_3)(2a_2 + a_4)] + \mu[(3c_1 + c_3)(2b_2 + b_4) - (3b_1 + b_3)(2c_2 + c_4)]$$
$$= \lambda \varphi(\vec{v}_1, \vec{v}_2) + \mu \varphi(\vec{w}, \vec{v}_2)$$

which shows linearity in the first coordinate. The proof of linearity in the second coordinate is analogous.

For antisymmetry,

$$\varphi(\vec{v}_2, \vec{v}_1) = (3b_1 + b_3)(2a_2 + a_4) - (3a_1 + a_3)(2b_2 + b_4)$$
$$= -[(3a_1 + a_3)(2b_2 + b_4) - (3b_1 + b_3)(2a_2 + a_4)] = -\varphi(\vec{v}_1, \vec{v}_2)$$

where the middle equality comes from an elementary comparison of the terms on each side after distributing.

(b). If we notice that

$$\varphi(\vec{v}_1, \vec{v}_2) = 6(a_1 b_2 - b_1 a_2) - 2(a_2 b_3 - a_3 b_2) + 3(a_1 b_4 - a_4 b_1) + (a_3 b_4 - b_3 a_4)$$

Then it is clear that $\varphi = 6dx_1 \wedge dx_2 - 2dx_2 \wedge dx_3 + 3dx_1 \wedge dx_4 + dx_3 \wedge dx_4$ by definition.

(c). Since any wedge product of forms with dx_i occurring twice is 0, the only term that does not disappear in the wedge product is $6dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_3$. This is the same as $-6dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. By definition $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$ is the same as the determinant. Thus $\lambda = -6$.

3 (20 pts). Let S be the surface given by $z = \sqrt{(x^2+y^2)}$ for $0 < z < 1$.

(a) (10 pts). Find a parameterization of this surface and calculate its surface area.

(b) (10 pts). Let $f(x,y,z) = z - \sqrt{(x^2+y^2)}$, so that S is the set of points where f is zero. Orient S using the gradient vector $\vec{\nabla}_f$, in words using the formula $\Omega_{\vec{x}}(\vec{v_1}, \vec{v_2}) = \operatorname{sgn}\det(\vec{\nabla}_f, \vec{v}_1, \vec{v}_2)$.

Find $\int_S \varphi$ for the 2-form field $\varphi = z\, dx \wedge dy$.

Solution.

(a). This is a cone for which cylindrical coordinates are helpful for parametrizing. We find the parameterization $\gamma : U \to S$ given by $\gamma\left(\begin{bmatrix}\theta\\ r\end{bmatrix}\right) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r\end{bmatrix}$, where

$$U = \left\{\begin{bmatrix}\theta\\ r\end{bmatrix} |0 \le \theta \le 2\pi, 0 \le r \le 1\right\}.$$

We calculate the derivative:

$$D\gamma(\theta, r) = \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \\ 0 & 1 \end{bmatrix}.$$

The surface area is by definition

$$\operatorname{vol}_2 S = \int_U \sqrt{\det([D\gamma(\vec{u})]^T[D\gamma(\vec{u})])}|d^2\vec{u}|.$$

We have

$$(D\gamma)^T D\gamma = \begin{bmatrix} -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{bmatrix} \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r^2 & 0 \\ 0 & 2 \end{bmatrix}$$

and so $\sqrt{\det([D\gamma(\vec{u})]^T[D\gamma(\vec{u})])} = \sqrt{2r^2} = \sqrt{2}r$.

Then

$$\operatorname{vol}_2(S) = \int_0^{2\pi}\int_0^1 \sqrt{2}r\,dr\,d\theta = \sqrt{2}\pi.$$

(b). We already have a parametrization from part (a). We check to see if it preserves the specified orientation on S. Note that $\vec{\nabla}_f = \begin{bmatrix} -2x(x^2+y^2)^{-1/2} \\ -2y(x^2+y^2)^{-1/2} \\ 1 \end{bmatrix}$. We calculate

$$\Omega_{\gamma(\vec{\theta},r)}(D_1\gamma(\theta,r), D_2\gamma(\theta,r)) = \det \begin{bmatrix} -2\cos\theta & -r\sin\theta & \cos\theta \\ -2\sin\theta & r\cos\theta & \sin\theta \\ 1 & 0 & 1 \end{bmatrix} = -3r.$$

This is negative for all points on S, so the parametrization reverses orientation. Rather than changing parametrization, we just adjust the sign to account for this. Now

$$\int_S z\,dx \wedge dy = -\int_{[\gamma(U)]} z\,dx \wedge dy = -\int_0^{2\pi}\int_0^1 r \det \begin{bmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{bmatrix} drd\theta$$

$$= -\int_0^{2\pi}\int_0^1 r(-r) = \int_0^{2\pi}\int_0^1 r^2 = 2\pi/3.$$

1. Some definitions and formulas

Definition 1.1. *Suppose that $M \subseteq \mathbb{R}^n$ is a k-dimensional manifold. Let $\gamma : U \to \mathbb{R}^n$ be a relaxed parametrization of M, where X is the set of bad points. Then*

$$\operatorname{vol}_k M = \int_{U-X} \sqrt{\det([D\gamma(\vec{u})]^T [D\gamma(\vec{u})])} |d^k \vec{u}|.$$

Definition 1.2. *let $U \subseteq \mathbb{R}^k$ be a bounded open set with $\operatorname{vol}_k \partial U = 0$. Let V be an open subset of $\mathbb{R}^n$ and let $\gamma : U \to \mathbb{R}^n$ be a C^1 mapping with $\gamma(U) \subseteq V$. Let φ be a k-form field defined on V. Then the integral of φ over $[\gamma(U)]$ is*

$$\int_{[\gamma(U)]} \varphi = \int_U \varphi(P_{\gamma(\vec{u})}(\vec{D}_1 \gamma(\vec{u}), \ldots, \vec{D}_k \gamma(\vec{u}))) |d^k \vec{u}|.$$

Definition 1.3. *Let $M \subseteq \mathbb{R}^n$ be a k-dimensional oriented manifold, φ a k-form field on a neighborhood of M, and $\gamma : U \to \mathbb{R}^n$ an orientation-preserving parametrization of M. Then*

$$\int_M \varphi = \int_{[\gamma(U)]} \varphi$$

as defined in the previous definition.