MATH 31CH SPRING 2017 MIDTERM 1

Instructions: Justify all of your answers, and show your work. You may quote basic theorems proved in the textbook or in class, unless the point of the problem is to reproduce the proof of such a theorem, or the problem tells you not to. Do not quote the results of homework exercises.

1 (10 pts). Let $A = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix}$.

Let $\vec{v}_1, \ldots, \vec{v}_4 \in \mathbb{R}^4$ be the four columns of A. Let $P = P(\vec{v}_1, \ldots, \vec{v}_4)$ be the 4-parallelogram spanned be these four vectors.

(a). Find $\mathrm{vol}_4 P$.

(b). Find $\int_{\mathbb{R}^4} \mathbf{1}_P \; x_3 \; |d^4\vec{x}|$.

2 (10 pts). Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables. Suppose that f is continuous. Let $A \subseteq \mathbb{R}^n$ be a bounded, closed subset. Prove that $\mathbf{1}_A \; f(\vec{x})$ is an integrable function on $\mathbb{R}^n$. (Hint: A continuous function on a bounded closed subset is *uniformly continuous*: for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\vec{x}, \vec{w} \in A$ satisfy $|\vec{x} - \vec{w}| < \delta$, then $|f(\vec{x}) - f(\vec{w})| < \epsilon$.)

3 (10 pts). Let A be the following region of $\mathbb{R}^3$: $A = \{(x, y, z) | 0 \le x, 0 \le y, 0 \le z, x^2 + y^2 + z^2 \le 1\}$.

Evaluate the integral $\int_{\mathbb{R}^3} \mathbf{1}_A \; (x^2 + y^2 + z^2)^{1/2} \; |dx||dy||dz|$.

4 (10 pts). Evaluate the integral

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x} \; dx \; dy.$$

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Some useful theorems:

Theorem 0.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, and $f : \mathbb{R}^n \to \mathbb{R}$ an integrable function. Then $f \circ T$ is integrable, and*

$$\int_{\mathbb{R}^n} f(\vec{y})|d^n\vec{y}| = |\det T| \int_{\mathbb{R}^n} f(T(\vec{x}))|d^n\vec{x}|.$$

Theorem 0.2. *Let X be a compact subset of $\mathbb{R}^n$ with boundary ∂X of volume 0; let $U \subset \mathbb{R}^n$ be an open set containing X. Let $\Phi : U \to \mathbb{R}^n$ be a C^1 mapping that is injective on $(X - \partial X)$ and has Lipschitz derivative, with $[D\Phi(\vec{x})]$ invertible at every $\vec{x} \in (X - \partial X)$. Set $Y = \Phi(X)$.*

Then if $f : Y \to \mathbb{R}$ is integrable, $(f \circ \Phi)|\det[D\Phi]|$ is integrable on X and

$$\int_Y f(\vec{y})|d^n\vec{y}| = \int_X (f \circ \Phi)(\vec{x})|\det D\Phi(\vec{x})||d^n\vec{x}|.$$

Lemma 0.3. *(Spherical coordinates) If*

$$S : \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi \end{pmatrix}$$

then $|\det[DS]| = r^2\cos\varphi$.

Lemma 0.4. *(Cylindrical coordinates) If*

$$S : \begin{pmatrix} r \\ \theta \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

then $|\det[DS]| = r$.