MATH 31BH WINTER 2018 MIDTERM 1— SAMPLE SOLUTIONS

1 (15 pts).

(a) (10 pts). Let U and V be open subsets of $\mathbb{R}^n$. Prove that $U \cap V$ is also open in $\mathbb{R}^n$.

(b) (5 pts). For each $k \geq 1$ suppose that U_k is an open subset of $\mathbb{R}^n$. Let Y be the intersection of all of these sets, in other words

$$Y = \bigcap_{k=1}^{\infty} U_k = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \in U_k \text{ for all } k \geq 1\}.$$

Must Y be open in $\mathbb{R}^n$? Prove or give a counterexample.

Solution.

(a) Let $\vec{x} \in U \cap V$. We need to find $\delta > 0$ such that $B(\vec{x}, \delta) \subseteq U \cap V$, where $B(\vec{x}, \delta) = \{\vec{a} \in \mathbb{R}^n \mid \|\vec{x} - \vec{a}\| < \delta\}$ is the open ball of radius δ around $\vec{x}$.

Since U is open and $\vec{x} \in U$, there must be $\delta_1 > 0$ such that $B(\vec{x}, \delta_1) \subseteq U$. Similarly, since V is open and $\vec{x} \in V$, there must be $\delta_2 > 0$ such that $B(\vec{x}, \delta_2) \subseteq V$. Now let $\delta = \min(\delta_1, \delta_2)$ and note that $\delta > 0$ still. Then $B(\vec{x}, \delta) \subseteq B(\vec{x}, \delta_1) \subseteq U$ and $B(\vec{x}, \delta) \subseteq B(\vec{x}, \delta_2) \subseteq V$. Hence $B(\vec{x}, \delta) \subseteq U \cap V$ as required.

(b) The set Y is not open in general. Take $n = 1$ and let $U_k = (-1/k, 1/k)$. Since U_k is an open interval, we know it is open in $\mathbb{R}$. However, we claim that $\bigcap_{k=1}^{\infty} U_k = \{0\}$. To see this, first note that it is obvious that $0 \in U_k$ for all k. On the other hand, if $a \neq 0$ we can find a natural number k such that $1/k < |a|$. Then $a \notin U_k$ and so $a \notin \bigcap_{k=1}^{\infty} U_k$. This proves the claim. Now it is easy to see that $\{0\}$ is not an open set in $\mathbb{R}$: No matter what $\delta > 0$ we pick, $B(0, \delta) = (-\delta, \delta)$ will contain nonzero numbers and will not be contained in $\{0\}$.

Remark 0.1. *Many students incorrectly claimed Y was open as an application of part (1). Some students proved correctly by induction that $U_1 \cap U_2 \cap \cdots \cap U_m$ is open for any natural number m. However, in general none of the finite intersections $U_1 \cap U_2 \cap \cdots \cap U_m$ is equal to the infinite intersection $\bigcap_{k=1}^{\infty} U_k$, so this does not settle anything.*

Another incorrect proof in the style of (1) is to find δ_k for each k such that $B(\vec{x}, \delta_k) \subseteq U_k$ and then let $\delta = \min(\delta_k)$. First, one should be careful with minimums: the minimum of a finite set of real numbers is defined but the minimum of an infinite set is not. However, you could let δ be the infimum *(greatest lower bound) of $\{\delta_k | k \geq 1\}$. The problem is that it is possible then that $\delta = 0$, so $B(\vec{x}, \delta)$ is not an open ball. That is precisely what happens in the explicit example above, where in fact $U_k = (-1/k, 1/k) = B(0, 1/k)$, and the infimum of $\{1/k \mid k \geq 1\}$ is 0.*

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2 (15 pts). Suppose that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix $A = [T]$, where A is an $m \times n$ matrix. Thus $T(\vec{x}) = A\vec{x}$.

(a) (5 pts). Show that for all vectors $\vec{x} \in \mathbb{R}^n$, $\|A\vec{x}\| \le c\|\vec{x}\|$, where $c = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$.

(b) (10 pts). Using part (a), prove that T is a continuous function.

Solution.

(a). Let $A_i \in \mathbb{R}^n$ be the transpose of the ith row of the matrix A. Then the ith entry of $A\vec{x}$ is equal to the dot product $A_i \cdot \vec{x}$. By Cauchy-Schwarz, $|A_i \cdot \vec{x}| \le \|A_i\|\|\vec{x}\|$. Here, $\|A_i\| = \sqrt{a_{i1}^2 + \cdots + a_{in}^2}$.

Thus

$$\|A\vec{x}\| = \sqrt{(A_1 \cdot \vec{x})^2 + \cdots + (A_m \cdot \vec{x})^2} \le \sqrt{(\|A_1\|\|\vec{x}\|)^2 + \cdots + (\|A_m\|\|\vec{x}\|)^2}$$

$$= \sqrt{\|A_1\|^2 + \cdots + \|A_m\|^2}\|\vec{x}\| = \sqrt{\sum_{j=1}^n a_{1j}^2 + \cdots + \sum_{j=1}^n a_{mj}^2}\|\vec{x}\| = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}\|\vec{x}\|$$

as needed.

(b). Let $\vec{a} \in \mathbb{R}^n$. We prove that T is continuous at the point $\vec{a}$. Given $\epsilon > 0$, we need to find $\delta > 0$ such that if $\|\vec{x} - \vec{a}\| < \delta$ then $\left\|T(\vec{x}) - T(\vec{a})\right\| < \epsilon$.

Let c be the constant found in part (a). If $c = 0$ then necessarily A is the zero matrix and so $T(\vec{x}) = \vec{0}$ for all $\vec{x}$; that is, T is the zero linear transformation. In this case given $\epsilon > 0$ we can choose any $\delta > 0$ we please. Then $\left\|T(\vec{x}) - T(\vec{a})\right\| = \left\|\vec{0} - \vec{0}\right\| = 0 < \epsilon$ for all $\vec{x} \in \mathbb{R}^n$, so certainly for $\vec{x}$ with $\|\vec{x} - \vec{a}\| < \delta$.

Thus now assume that $c \neq 0$. Given $\epsilon > 0$, we take $\delta = \epsilon/c$. Note that since T is linear, we have $T(\vec{x}) - T(\vec{a}) = T(\vec{x} - \vec{a})$. Thus if $\|\vec{x} - \vec{a}\| < \delta$, we have

$$\left\|T(\vec{x}) - T(\vec{a})\right\| = \left\|T(\vec{x} - \vec{a})\right\| \le c\|\vec{x} - \vec{a}\| < c\delta = \epsilon,$$

where we have used part (a) in the first inequality. This proves continuity at the point $\vec{a}$.

3 (10 pts). Write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Let $f : (\mathbb{R}^2 - \{\vec{0}\}) \to \mathbb{R}$ be given by $f(\vec{x}) = \dfrac{x_1}{\sqrt{x_1^2 + x_2^2}}$.

Does $\lim_{\vec{x} \to \vec{0}} f(\vec{x})$ exist? Justify your answer carefully using the (ϵ, δ)-definition of the limit.

Solution.

The limit does not exist. The problem can be seen by looking along lines approaching the origin. When $x_2 = 0$ and $x_1 > 0$, we have $f(\vec{x}) = x_1/\sqrt{x_1^2} = 1$. When $x_1 = 0$ and $x_2 \neq 0$, we have $f(\vec{x}) = 0$. Points of both types lie in any ball of radius δ around the origin, and this prevents the limit from existing.

More formally, we write the proof as follows. Suppose that $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = b$. Let $\epsilon = 1/2$. Then there must exist $\delta > 0$ such that if $\left\|\vec{x} - \vec{0}\right\| = \|\vec{x}\| < \delta$, then $|f(\vec{x}) - b| < \epsilon = 1/2$. Now $B(\vec{0}, \delta)$ contains both the point $\vec{x} = \begin{pmatrix} \delta/2 \\ 0 \end{pmatrix}$ for which $f(\vec{x}) = 1$, and the point $\vec{x}' = \begin{pmatrix} 0 \\ \delta/2 \end{pmatrix}$ for which $f(\vec{x}') = 0$. Thus in particular we must have both $|0 - b| < 1/2$ and $|1 - b| < 1/2$. But then $1 = |1| = |b + (1 - b)| \leq |b| + |1 - b| < 1/2 + 1/2 = 1$, a contradiction. Thus the limit does not exist.

Remark 0.2. *An alternative approach is to make the intuition coming from approaching the origin through two different lines more formal. A good way to do this is to use the notion of convergent sequence. The following lemma is standard— it is not quite stated in the book but the proof is essentially the same as the proof of Proposition 3.6. You are welcome to use it from now on.*

Lemma 0.3. *Let $f : U - \{a\} \to \mathbb{R}^m$ be a function, where $U \subseteq \mathbb{R}^n$ is open. Then $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{b}$ if and only if for all sequences $\{\vec{x}_k\}$ of points in $U - \{a\}$ such that $\lim_{k\to\infty} \vec{x}_k = a$, we have $\lim_{k\to\infty} f(\vec{x}_k) = \vec{b}$.*

If we assume this lemma, then we note that $\{\vec{x}_k = \begin{bmatrix} 1/k \\ 0 \end{bmatrix}\}$ and $\{\vec{x}'_k = \begin{bmatrix} 0 \\ 1/k \end{bmatrix}\}$ are both sequences with $\lim_{k\to\infty} \vec{x}_k = \vec{0}$ and $\lim_{k\to\infty} \vec{x}'_k = \vec{0}$. However $f(\vec{x}_k) = 1$ for all k and $f(\vec{x}'_k) = 0$ for all k, so $\lim_{k\to\infty} f(\vec{x}_k) = 1$ and $\lim_{k\to\infty} f(\vec{x}'_k) = 0$. By the lemma, $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ cannot exist, as it would have to equal both 0 and 1.