MATH 31AH Midterm 2 Solutions

1) a) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Recalling that reduced echelon form requires that the only non-zero entry in a column which contains a pivot is the pivot itself, we find that the reduced echelon form of A is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As the rank of a matrix is the number of pivots, we see that the rank of A is 2.

b) We refer to our answer in part (a). The columns which contain pivot variables in the reduced echelon form of A form a basis for $C(A)$. Likewise, the rows which contain pivots in the reduced echelon form of A form a basis for $R(A)$. We deduce then that

$$C(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$R(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

Looking at the third column in the reduced echelon form of A tells us that the third column of A is twice the first column vector minus the second column vector. Hence, we have

$$N(A) = \text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

2) We row reduce the augment matrix

$$\begin{bmatrix} 2 & 3 & 4 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 2 & 1 \\ 0 & 1 & 0 & 5 & -8 & -6 \\ 0 & 0 & 1 & -3 & 5 & 4 \end{bmatrix}.$$

So we deduce that

$$A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 5 & -8 & -6 \\ -3 & 5 & 4 \end{bmatrix}.$$

3) a) There can be no such matrix, for if $Ax = b_3$ has infinitely many solutions, we know $\text{rank}(A) < n$. But to say that $Ax = b_2$ has a unique solution is to say that $\text{rank}(A) = n$.

b) Such a matrix does exist. As an example, consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

c) No such matrix exists. If $Ax = 0$ were to admit a unique solution, namely the trivial solution $x = 0$, then we would have that $\text{rank}(A) = n$. However, for any $A \in \mathbb{R}^{m\times n}$ we have $\text{rank}(A) \le \min\{m, n\} = n$.

4) Suppose there exist $B, C \in \mathbb{R}^{m\times n}$ such that $AB = I_m$ and $CA = I_n$. From the first equality, we see that for any $b \in \mathbb{R}^m$, $x = Bb$ satisfies $Ax = b$. Therefore, $\text{rank}(A) = m$. On the other hand, if $Ax = 0$, then multiplying by C on both sides yields $x = 0$. In other words, $Ax = 0$ only has $x = 0$ as a solution. Consequentially $\text{rank}(A) = n$. From this we see that $m = n$. Moreover, multiplying $AB = I_m$ by C on both sides yields $B = C$. Therefore $B = C = A^{-1}$, so A is invertible.
Conversely, suppose that $m = n$ and A is invertible. Then setting $B = C = A^{-1}$ yields the desired conclusion.

5) a) We first show linear independence. Suppose we have scalars $c_1, \dots, c_n \in \mathbb{R}$ such that

$$c_1 v_1 + \dots + c_n v_n = 0. \tag{1}$$

If we dot each side of the equation by v_i, the mutual orthogonality tells us that

$$c_i \|v_i\|^2 = 0.$$

Since each of the v_i are non-zero, we conclude that c_i is zero. As this is true for $i \in \{1, \dots, n\}$, we see that the collection $\{v_1, \dots, v_n\}$ is linearly independent.
To show that the collection spans R^n, we note that the dimension of R^n is n. Since $V := \text{Span}\{v_1, \dots, v_n\}$ satisfies $dim(V) = n$ and $V \subseteq \mathbb{R}^n$, it must be that $V = \mathbb{R}^n$.

b) We claim that $\ker T = \text{Span}\{v_2, \dots, v_n\}$. By mutual orthogonality, we certainly have $\text{Span}\{v_2, \dots, v_n\} \subseteq \ker T$. To show the reverse inclusion, suppose we have $x \in \ker T$. As $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$, we know there exist scalars $a_1, \dots, a_n$ such that $x = a_1 v_1 + \dots + a_n v_n$. Then

$$T(x) = 0 \implies a_1 T(v_1) + \dots + a_n T(v_n) = 0 \implies a_1 T(v_1) = a_1 \|v_1\|^2 = 0 \implies a_1 = 0.$$

So $x \in \text{Span}\{v_2, \dots, v_n\}$, as desired.

6) a) First, we check that the zero polynomial is in O_k. Let $p(x) = 0$. Then $p(x) = 0 = -0 = -p(-x)$, so $p \in O_k$.
Next, we check that O_k is closed under addition. Suppose $f, g \in O_k$. Then

$$-(f+g)(-x) = -f(-x) - g(-x) = f(x) + g(x) = (f+g)(x)$$

so $f + g \in O_k$.
Lastly we check closure under scalar multiplication. Suppose that $c \in \mathbb{R}$ and $f \in O_k$. Then

$$-(cf)(x) = -cf(-x) = c(-f)(-x) = cf(x) = (cf)(x).$$

Therefore, O_k is indeed a subspace.

b) We nominate $\{x, x^3, \dots, x^\ell\}$ as a basis for O_k, where $\ell = k-1$ if k is even, and $\ell = k$ if k is odd. To begin, it is clear that this collection is linearly independent, as this collection is a subset of the monomials $\{1, x, x^2, \dots, x^k\}$ which are linearly independent. To show that they span, suppose $f \in O_k$. Since the monomials are a basis for P_k, we may write

$$f(x) = a_0 + a_1 x + \dots + a_k x^k$$

for some $a_0, \dots, a_k \in \mathbb{R}$. Since f is odd, we must have $f(x) = -f(-x)$. In other words,

$$a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k = -a_0 + a_1 x - a_2 x^2 + \dots + (-1)^k a_k x^k.$$

Comparing term by term, we see that in order for this equality to be true it must be that $a_2 = -a_2, \dots a_{\lceil k/2 \rceil} = -a_{\lceil k/2 \rceil}$, which is only true if the aforementioned terms are all zero. Therefore, we have that

$$f(x) = a_1 x + a_3 x^3 + \dots + a_\ell x^\ell.$$

If k is even, this means the dimension of O_k is $\frac{k}{2}$. Otherwise, the dimension is $\frac{k+1}{2}$.