MATH 31AH FALL 2017 MIDTERM 1: SAMPLE SOLUTIONS

1 (10 pts).

(a) Find the standard matrix $A = [T]$ of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by projection onto the line spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. You do not need to prove your answer is correct, but you must show your calculations.

(b) Find the standard matrix $B = [S]$ of the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection across the line spanned by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. You do not need to prove your answer is correct, but you must show your calculations.

(c) Are the linear transformations $S \circ T$ and $T \circ S$ the same? Briefly justify your answer.

Solution.

[Note that parts (a) and (b) were a webwork problem, but with easier vectors chosen than the vectors the webwork typically randomly picked].

(a). Let $\vec{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and let $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as usual. Then

$$\operatorname{proj}_{\vec{y}} \vec{e_1} = \frac{\vec{y} \cdot \vec{e_1}}{\|\vec{y}\|^2} \vec{y} = \frac{1}{1^2 + 1^2} \vec{y} = \frac{1}{2} \vec{y} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$

A similar calculation gives $\operatorname{proj}_{\vec{y}} \vec{e_2} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$. Since $[T]$ is the matrix with columns $T(\vec{e_1})$ and $T(\vec{e_2})$, we have

$$[T] = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

(Most students did the problem this way, since this was the best method in the webwork where the vector $\vec{y}$ was random. In this problem, it is also fine to draw a picture to visually calculate what the projections of $\vec{e_1}$ and $\vec{e_2}$ are, which works since the line through $\vec{y}$ makes a nice 45 degree angle with the axes.)

(b). Some students recalled formulas for the matrix of a reflection in terms of the angle the reflection line makes with the x-axis, that were obtained in a homework problem.

It is also possible to obtain the reflection of a vector $\vec{x}$ about the line through $\vec{y}$ by writing $\vec{x} = \vec{x}^{\perp} + \vec{x}^{\|}$, where $\vec{x}^{\|}$ is the projection of $\vec{x}$ onto $\vec{y}$. Then the reflection of $\vec{x}$ about the line through $\vec{y}$ is $\vec{x} - 2\vec{x}^{\perp}$. (draw a picture to convince yourself).

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The easiest way to do part (b) in this case, since the vector $\vec{y}$ is so simple, is just to draw a picture and visually calculate the reflections of $\vec{e_1}$ and $\vec{e_2}$ about this line. From the picture one sees that $S(\vec{e_1}) = -\vec{e_2}$ and $S(\vec{e_2}) = -\vec{e_1}$. Thus

$$[S] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(c). This was meant to refer to the particular linear transformations S and T in parts (a) and (b); I'm sorry if that was unclear. (Usually, letters are meant to have a consistent meaning throughout all parts of a problem).

Then, since $[S \circ T] = [S][T]$ and $[T \circ S] = [T][S]$, we can tell if the two compositions are the same by computing the two products of their standard matrices. Since

$$[S][T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}$$

and

$$[T][S] = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix},$$

the two compositions are the same.

An alternative method was to argue geometrically that projecting onto the line through $[1,1]$ and then reflecting about the line through $[1,-1]$ gives the same result as first reflecting through the line through $[1,-1]$ and then projecting onto the line through $[1,1]$ (try it).

If you interpreted this problem to mean "given two arbitrary linear transformations S and T, does $S \circ T = T \circ S$?" then to receive full credit you had to give an example where this fails, not just state that it is not true in general (since the problem asked for a justification).

2 (10 pts).

(a) Find a vector orthogonal to the plane in $\mathbb{R}^3$ which goes through the three points

$$\begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}, \text{and} \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}.$$

You do not need to prove your answer is correct, but show your calculations.

(b) Find the equation of the plane that goes through the three points given in (a), writing the equation in the form $ax + by + cz = d$. You do not need to prove your answer is correct, but show your calculations.

Solution.

[Note that this was a webwork problem, though when you did it you probably had different numbers].

(a). We need to find two vectors that lie on the plane. For this we take two differences of the given points, for example $\vec{v_1} = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ and $\vec{v_2} = \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then the quickest way to find a vector orthogonal to the plane through two (non-parallel) vectors is to calculate their cross product:

$$\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \det \begin{bmatrix} i & 1 & 1 \\ j & -2 & 1 \\ k & -1 & 1 \end{bmatrix} = -2i + k - j - (-2k - i + j) = -i - 2j + 3k = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}.$$

It is also possible to find such a vector by setting up a system of linear equations and solving it, but the cross product is faster.

(b). As explained in Example 2 of section 1.5 (page 49 of the text), any plane orthogonal to the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ will have the formula $ax + by + cz = d$ for some d. Thus the plane we are looking for has the formula $-x - 2y + 3z = d$. Since all three given points are supposed to lie on this plane, we can solve for d by plugging in any of these three points, for example $-(3) - 2(-3) + 3(5) = d$ so $d = 18$. Thus the answer is $-x - 2y + 3z = 18$.

Again, this could also be done by setting up a system of equations to solve for all of a, b, c, d but this is lengthy as it does not use the information you got in part (a).

3 (10 pts).

(a) Use vector methods to prove that the diagonals of a parallelogram are orthogonal if and only if the parallelogram is a rhombus (that is, has all sides of equal length).

(b) Use vector methods to prove that the diagonals of a parallelogram bisect the vertex angles if and only if the parallelogram is a rhombus.

Solution.

(a). Let the vertices of the parallelogram be A, B, C, D as we go around the parallelogram clockwise. Let $\vec{x} = \vec{AB}$ and $\vec{y} = \vec{AD}$. (Note that this renaming makes the notation in the problem much simpler). Then the two diagonals of the parallelogram are $\vec{AC} = \vec{x} + \vec{y}$ and $\vec{BD} = \vec{y} - \vec{x}$.

Suppose the diagonals of the parallelogram are orthogonal. This is equivalent to $(\vec{x} + \vec{y}) \cdot (\vec{y} - \vec{x}) = 0$. But using the properties of the dot product we now have

$$(\vec{x} + \vec{y}) \cdot (\vec{y} - \vec{x}) = \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} - \vec{y} \cdot \vec{x} - \vec{x} \cdot \vec{x} = \|\vec{y}\|^2 - \|\vec{x}\|^2.$$

We see that $\|\vec{y}\|^2 = \|\vec{x}\|^2$, and since lengths are nonnegative numbers, taking the square root of both sides yields $\|\vec{y}\| = \|\vec{x}\|$. This shows that the adjacent sides AB and AD have the same length. But since opposite sides in any parallelogram have the same length, it follows that all sides of the parallelogram have the same length.

Conversely, if the parallelogram is a rhombus, then the sides AB and AD have the same length, so $\|\vec{y}\| = \|\vec{x}\|$, and squaring gives $\|\vec{y}\|^2 = \|\vec{x}\|^2$. The same equation displayed above still holds, so this implies that $0 = (\vec{x} + \vec{y}) \cdot (\vec{y} - \vec{x})$, which says that the diagonals of the parallelogram are orthogonal.

(b). Again let the parallelogram have vertices named A, B, C, D as we go around clockwise and let $\vec{x} = \vec{AB}$ and $\vec{y} = \vec{AD}$. Consider the angle at vertex A. The diagonal that goes through this vertex is $\vec{AC} = \vec{x} + \vec{y}$. Let θ_1 be the angle between $\vec{x}$ and $\vec{x} + \vec{y}$ and let θ_2 be the angle between $\vec{x} + \vec{y}$ and $\vec{y}$. From the formula for the dot product we get

$$\vec{x} \cdot (\vec{x} + \vec{y}) = \|\vec{x}\| \|\vec{x} + \vec{y}\| \cos \theta_1$$

and so

$$\cos \theta_1 = \frac{\vec{x} \cdot (\vec{x} + \vec{y})}{\|\vec{x}\| \|\vec{x} + \vec{y}\|} = \frac{\|\vec{x}\|^2 + \vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{x} + \vec{y}\|}.$$

A similar calculation gives

$$\cos \theta_2 = \frac{\|\vec{y}\|^2 + \vec{x} \cdot \vec{y}}{\|\vec{y}\| \|\vec{x} + \vec{y}\|}.$$

Now suppose that the parallelogram is a rhombus, so that $\|\vec{x}\| = \|\vec{y}\|$. Plugging in this equality into the equations above we get that $\cos \theta_1 = \cos \theta_2$. Now the angle between two vectors is taken in the range $[0, \pi]$. For angles in that range, two angles have the same cosine if and only if they are the same. Thus $\theta_1 = \theta_2$ and the diagonal AC bisects the angle at vertex A.

Now there was nothing special about the vertex A in this argument. Thus assuming the parallelogram is a rhombus, the same argument applied at each vertex shows that the diagonal at that vertex bisects the angle at that vertex.

Conversely, suppose that the diagonal AC bisects the angle at vertex A. Then $\theta_1 = \theta_2$ and so certainly $\cos \theta_1 = \cos \theta_2$. Using the formulas above we get an equality

$$\frac{\|\vec{y}\|^2 + \vec{x} \cdot \vec{y}}{\|\vec{y}\| \|\vec{x} + \vec{y}\|} = \frac{\|\vec{x}\|^2 + \vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{x} + \vec{y}\|}.$$

Cancelling the term $\|\vec{x} + \vec{y}\|$ (note that this is the length of a diagonal of a parallelogram and so cannot be 0) and clearing denominators gives $\|\vec{x}\|(\|\vec{y}\|^2 + \vec{x} \cdot \vec{y}) = \|\vec{y}\|(\|\vec{x}\|^2 + \vec{x} \cdot \vec{y})$. After some algebraic manipulation we get $\|\vec{x}\| \|\vec{y}\|(\|\vec{y}\| - \|\vec{x}\|) = (\|\vec{y}\| - \|\vec{x}\|)\vec{x} \cdot \vec{y}$.

Now if $\|\vec{y}\| - \|\vec{x}\|$ is not zero, we can divide both sides by this quantity to obtain $\|\vec{x}\| \|\vec{y}\| = \vec{x} \cdot \vec{y}$. Note that this says that the angle ϕ between the vectors $\vec{x}$ and $\vec{y}$ has $\cos \phi = 1$, in other words $\phi = 0$. This is impossible for a parallelogram. Thus we must have $\|\vec{y}\| - \|\vec{x}\| = 0$, and so $\|\vec{y}\| = \|\vec{x}\|$. This shows that the parallelogram has two adjacent sides of equal length, and as in the proof of part (a), this implies that all sides have the same length.

4 (10 pts).

Suppose that U and V are subspaces of $\mathbb{R}^n$. Recall that we define

$$U + V = \{\vec{x} \in \mathbb{R}^n \,|\, \vec{x} = \vec{u} + \vec{v} \text{ for some } \vec{u} \in U \text{ and } \vec{v} \in V\}.$$

Prove that $(U + V)^{\perp} = U^{\perp} \cap V^{\perp}$.

Solution.

[Note that this was a homework problem verbatim, and it was one of the problems that was graded. If you did not pick up your graded homework 2 assignment and view the comments there, you may have replicated any mistakes you made on your exam. Remember that it is important to view the feedback you obtain from the homework grader.]

Suppose first that $\vec{x} \in (U + V)^{\perp}$. This means that $\vec{x} \cdot (\vec{u} + \vec{v}) = 0$ for all $\vec{u} \in U$ and all $\vec{v} \in V$. Using the linearity of the dot product we get that $\vec{x} \cdot \vec{u} + \vec{x} \cdot \vec{v} = 0$ for all $\vec{u} \in U$ and $\vec{v} \in V$.

Since V is a subspace, we have $\vec{0} \in V$. Apply the statement above to an arbitrary $\vec{u} \in U$ and the vector $\vec{v} = \vec{0} \in V$. We obtain $\vec{x} \cdot \vec{u} + \vec{x} \cdot \vec{0} = \vec{x} \cdot \vec{u} + 0 = 0$, for all $\vec{u} \in U$. Thus $\vec{x} \cdot \vec{u} = 0$ for all $\vec{u} \in U$, and this means that $\vec{x} \in U^{\perp}$ by definition. [This is the key part of the argument, which many of you missed. Note that just because we know a sum of two real numbers is 0, we cannot conclude that both numbers are 0 in general without some further reason.]

The same argument, switching the roles of U and V, shows that $\vec{x} \in V^{\perp}$. Thus $\vec{x} \in U^{\perp} \cap V^{\perp}$. We have proved that $(U + V)^{\perp} \subseteq U^{\perp} \cap V^{\perp}$.

Conversely, suppose that $\vec{x} \in U^{\perp} \cap V^{\perp}$. Then in particular $\vec{x} \in U^{\perp}$, and so $\vec{x} \cdot \vec{u} = 0$ for all $\vec{u} \in U$. Similarly, $\vec{x} \in V^{\perp}$, and so $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v} \in V$. Adding these equations we get $\vec{x} \cdot (\vec{u} + \vec{v}) = \vec{x} \cdot \vec{u} + \vec{x} + \vec{v} = 0 + 0 = 0$ for all $\vec{u} \in U$ and $\vec{v} \in V$. Sine $\vec{u} + \vec{v}$ is an arbitrary vector in $U + V$, we conclude that $\vec{x} \in (U + V)^{\perp}$. We have proved that $U^{\perp} \cap V^{\perp} \subseteq (U + V)^{\perp}$.

Since we have proved both inclusions, we conclude that $U^{\perp} \cap V^{\perp} = (U + V)^{\perp}$.

5 (10 pts).

Let A, B and C be $n \times n$ matrices. Let 0 be the $n \times n$ 0-matrix. For each of the following statements, if it is true for all $n \times n$ matrices A, B, and C with the indicated properties, prove it; otherwise give a counterexample.

(a) If $AB = BC$ and B is invertible, then $A = C$.
(b) If $AB = CB$ and B is invertible, then $A = C$.
(c) If $AB = 0$, then either $A = 0$ or $B = 0$.

Solution.

(a). [This was a homework problem verbatim]

This is false. One way to find a counterexample is the following. If B is invertible, then the equation $AB = BC$ leads to $B^{-1}AB = B^{-1}BC = C$ by multiplying on the left by B^{-1} on both sides. (Note that it is not valid to multiply on one side of the equation on the right and on the other side on the left, since multiplication of matrices is not commutative.) Thus if we pick any invertible matrix B and a random matrix A, then as long as $B^{-1}AB$ is different from A we have a counterexample.

For example pick $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is invertible with $B^{-1} = B$, and let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $B^{-1}AB = BAB = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$. Letting $C = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}$ we see that $AB = BC$ but $A \neq C$.

(b). This is true. Multiply the equation $AB = CB$ by B^{-1} on the right, obtaining $ABB^{-1} = CBB^{-1}$. Since $BB^{-1} = I_n$, we get $A = AI_n = CI_n = C$.

(c). This is false. There are many examples, which you can find by experimentation. One more systematic method is to recall that the columns of AB are given by the vectors $A\vec{b}_i$ where $\vec{b}_i$ are the columns of B. Thus choose a nonzero matrix A with a nonzero vector $\vec{v}$ such that $A\vec{v} = \vec{0}$. (From our more recent work in Chapter 4, we know this happens as long as A is a singular matrix.) For example, choose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then we can take both columns of B to be $\vec{v}$, i.e. $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. It follows that $AB = 0$ even though A and B are nonzero.

6 (15 pts).

Recall that an $n \times n$ matrix A is *orthogonal* if $A^T A = I_n$, where I_n is the $n \times n$ identity matrix. In this problem let $n = 2$.

(a) Show that if $\vec{a}_1$ and $\vec{a}_2$ are the two columns of the 2×2 matrix A, then A is orthogonal if and only if $\|\vec{a}_1\| = 1$, $\|\vec{a}_2\| = 1$, and $\vec{a}_1 \cdot \vec{a}_2 = 0$.

(b) Show that A is an orthogonal 2×2 matrix if and only if there are elements $a, c \in \mathbb{R}$ such that $a^2 + c^2 = 1$ and either $A = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$ or $A = \begin{pmatrix} a & c \\ c & -a \end{pmatrix}$.

(c) Show that if the 2×2 matrix A is orthogonal, then A is invertible and A^{-1} is also orthogonal.

(d) Show that if the 2×2 matrix A is orthogonal, then A^T is also orthogonal.

Solution.

[Most of this was homework problem(s), although in the homework (b) was phrased in terms of a choice of angle θ.]

(a). For any i and j, we can recognize $\vec{a_i} \cdot \vec{a_j}$ as the ijth spot of the matrix $A^T A$. Explicitly, If $A_{ij} = a_{ij}$ then

$$(A^T A)_{ij} = \sum_{k=1}^{2} (A^T)_{ik} A_{kj} = \sum_{k=1}^{2} A_{ki} A_{kj} = \sum_{k=1}^{2} a_{ki} a_{kj} = \vec{a_i} \cdot \vec{a_j}.$$

Since $A^T A = I_2$ by definition, the dot products of the columns of A have the values described.

(b). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be orthogonal. By part (a), the columns have length 1 and so $a^2 + c^2 = 1$ and $b^2 + d^2 = 1$. Also, the columns are orthogonal by part (a). Suppose that $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} a \\ c \end{bmatrix}$, and so $ax_1 + cx_2 = 0$. If $a \neq 0$, we get $x_1 = (-c/a)x_2$. If the vector $\vec{x}$ also has $\|\vec{x}\| = 1$, then $c^2 x_2^2/a^2 + x_2^2 = 1$ and so $x_2^2(c^2/a^2 + 1) = 1$. Since $a^2 + c^2 = 1$, this gives $x_2^2 = a^2$. So either $x_2 = a$ and $x_1 = -c$ or else $x_2 = -a$ and $x_1 = c$. This shows that once the first column of the matrix is chosen to be the vector $\begin{bmatrix} a \\ c \end{bmatrix}$ of length 1, the second column has to be either $\begin{bmatrix} -c \\ a \end{bmatrix}$ or $\begin{bmatrix} c \\ -a \end{bmatrix}$, which gives the two required forms. Although this argument assumed that $a \neq 0$, a similar argument assuming that $c \neq 0$ gives the same two possible forms for the second column. Since $a^2 + c^2 = 1$, either a or c has to be nonzero, so one of the two cases applies.

Conversely, if $a^2 + c^2 = 1$ then clearly for either of the two forms above, both columns have length 1 and the two columns are orthogonal, by direct calculation.

(c). Let A be orthogonal. Then by part (b) we know it has one of the two forms above. If $A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$ with $a^2 + c^2 = 1$, then $\det A = a^2 + c^2 = 1$. We showed in class that any 2×2 matrix with nonzero determinant has an inverse, and gave the formula for the inverse. Thus A is invertible and its inverse is $1/(\det A) \begin{bmatrix} a & c \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ -c & a \end{bmatrix}$. This matrix is again of the same form and so is orthogonal by part (b) again.

On the other hand, if $A = \begin{bmatrix} a & c \\ c & -a \end{bmatrix}$ with $a^2 + c^2 = 1$, then $\det A = -a^2 - c^2 = -1$. Again since this is nonzero we know that A is invertible with inverse $1/(\det A) \begin{bmatrix} -a & -c \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ c & -a \end{bmatrix}$, which again is orthogonal by part (b).

An alternative method which does not require the formula for the inverse of a 2×2 invertible matrix is to notice that $A^T A = I_2$ suggests that A^T should be the inverse of A. Thus one can compute AA^T in each of the two cases and show that it is also equal to I_2, proving that A is invertible with $A^{-1} = A^T$. Then one checks that A^T is also of one of the forms in (b) and so is orthogonal (see part (d) below).

(d). Considering the two forms in part (b), if A is of the first form then clearly A^T is of the same form (but with c replaced by $-c$). Thus A^T is also orthogonal by part (b). If A is of the second form, then $A^T = A$, so A is also orthogonal.

If one used the alternative method for (c), then this already proves that A^T is invertible with inverse A, and in part (c) one already proved that $A^T = A^{-1}$ was orthogonal.