

Math 207c Spring 2011 Homework exercises 1

Though you are not required to hand in these exercises, working through some of them (in addition to going over your lecture notes which is also very important) will help you get much more out of the course.

1. Let K be a field which is not necessarily algebraically closed. Let Q be the quiver with one vertex 1 and one loop $a : 1 \rightarrow 1$. Show that you can classify all simple representations and all indecomposable representations of Q . (In class we did the case where K is algebraically closed.) (Hint: $KQ \cong K[x]$ so this is equivalent to understanding finite-dimensional $K[x]$ -modules with certain properties. Use the theorems on modules over a PID.)

2. let Q be the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \dots \xrightarrow{a_{n-1}} n$$

For any $0 \leq i \leq j \leq n$, we have a representation of Q of the form

$$0 \longrightarrow \dots \longrightarrow 0 \longrightarrow K_i \xrightarrow{1} K \xrightarrow{1} \dots \xrightarrow{1} K_j \longrightarrow 0 \longrightarrow \dots \longrightarrow 0$$

Show that these are pairwise non-isomorphic indecomposable representations. Then show that every indecomposable representation of Q is isomorphic to one of these. Which of these indecomposables are projective representations and which are injective representations? Which are simple representations?

We will see later that the classification of indecomposable reps over this quiver also follows from a more general abstract theorem.

3. Let Q be the quiver with one vertex 1 and two loops $a : 1 \rightarrow 1$ and $b : 1 \rightarrow 1$. Let V be the following representation of Q : V_1 is a vector space with K -basis $v_1, \dots, v_n$ for some

$n \geq 1$; and the linear maps associated to the arrows are

$$V(a) : \begin{cases} v_i \mapsto v_{i+1} & 1 \leq i \leq n-1 \\ v_i \mapsto 0 & i = n \end{cases}, \quad V(b) : \begin{cases} v_i \mapsto v_{i-1} & 2 \leq i \leq n \\ v_i \mapsto 0 & i = 1 \end{cases}.$$

Show that V is a simple representation of Q . Conclude that Q has simple reps of all dimensions.

4. Fix any quiver Q and field K . Let Q^{op} be the opposite quiver, which has the same vertices as Q but replaces each arrow $a : i \rightarrow j$ in Q with an arrow $a^* : j \rightarrow i$. Let $\mathcal{C} = \text{rep}_K Q$ be the category of representations of Q and let $\mathcal{D} = \text{rep}_K Q^{op}$ be the category of representations of Q^{op} .

(a). Show that there is a contravariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ which does the following. Given a rep V of Q , so this has vector space V_i at vertex i and map $V(a) : V_i \rightarrow V_j$ for each arrow $a : i \rightarrow j$, $F(V)$ is the rep V^* which has $(V^*)_i = (V_i)^* = \text{Hom}_K(V_i, K)$ and $V^*(a^*) = V(a)^* : V_j^* \rightarrow V_i^*$, i.e. $V(a)^*(\phi) = \phi \circ V(a)$ for $\phi \in \text{Hom}_K(V_j, K)$. Show how to define the action of F in a natural way on morphisms to make F into a contravariant functor (i.e. F sends $f \in \text{Hom}_{\mathcal{C}}(V, W)$ to some $F(f) \in \text{Hom}_{\mathcal{D}}(W^*, V^*)$ so that $F(f \circ g) = F(g) \circ F(f)$ for morphisms $f : V \rightarrow W, g : W \rightarrow X$).

(b). Define a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ essentially in the same way, by using duals (note that $(Q^{op})^{op} = Q$.) Prove that F and G are inverse contravariant equivalences of categories.

Remark. This is the quiver-theoretic reflection of the duality between finite-dimensional left and right modules over the path algebra KQ . The duality above can be used explicitly to determine the injective representations of Q if we know the projective representations of Q^{op} .

5. (a). Let Q be a quiver without oriented cycles. Show that you can choose a labeling $\{1, 2, \dots, n\}$ of the vertex set of Q such that if there exists a path from i to j , then $i \leq j$.

(b). Let Q be a quiver with the property that for any vertices i, j , there exists at most one path (including trivial paths) from i to j . Note that this is a subset of those quivers with no oriented cycles (since the trivial path e_i goes from i to i , Q cannot contain loops.) Choose a labeling as in part (a).

Prove that KQ is isomorphic to a subalgebra of the lower triangular matrix algebra

$$T = \{(a_{ij}) \in M_n(K) \mid a_{ij} = 0 \text{ if } j > i\}.$$