

Math 207a Winter 2012 Homework exercises 2

Due Friday February 10

You are not expected to complete all exercises. Read the statements of them, and write up and hand in some selection of them which you find interesting. Many of these exercises are taken directly from the Goodearl-Warfield text, but we repeat them here since I do not require you to own the text, and besides I have an older edition of the text than the one currently available. Also, some exercises are more straightforward than others, though I generally won't assign extremely routine ones.

1 Prime and semiprime ideals

1. (a) Let R be a right noetherian ring, and let N be the prime radical of R . Prove that N is a nilpotent ideal, i.e. that there is $n \geq 1$ such that $N^n = 0$. (remember, as always, that N^n means the set of all finite sums of elements of the form $x_1 x_2 \cdots x_n$ where each $x_i \in N$.)

Hint: by the noetherian property, there exists an ideal maximal among nilpotent ideals in R . Show this ideal must be semiprime.

(b). Show the result above is false in general for non-noetherian rings, by considering the infinite product $\prod_{n \geq 1} \mathbb{Z}/2^n \mathbb{Z}$.

2. (a) Let R be a ring and fix $n \geq 1$. Prove that every ideal of $M_n(R)$ is of the form $M_n(I) = \{(a_{ij}) | a_{ij} \in I, \text{ for all } 1 \leq i, j \leq n\}$ for some ideal I of R . Conclude that if D is a division ring then $M_n(R)$ is a simple ring.

(b). Show if R is a semiprime ring then $M_n(R)$ is also semiprime.

3. If all prime ideals of a ring R are finitely generated (as left ideals), show that R is noetherian. (Hint: assume otherwise and find an ideal maximal among non-finitely generated ideals.)

2 The Jacobson radical

4. Find the Jacobson radical of a formal triangular matrix ring $\begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. (Your answer will be in terms of the Jacobson radicals of R and S .)

5. We didn't prove in class the part of the characterization of the Jacobson radical which says that if J_1 is the intersection of all max left ideals of R and J_2 is the intersection of all max right ideals of R , then $J_1 = J_2$. Prove this using the following sketch. (The proof is also in the text; so do this only if you have not read the proof.)

Let $x \in J_1$. Show that $(1-x)$ cannot be contained in any left ideal of R and so $z(1-x) = 1$ for some $z \in R$. Then $z = 1 + xz$; similarly, $1 + xz$ cannot be contained in any left ideal of R and so $z = 1 + xz$ also has a left inverse; but this forces $(1-x)z = 1$ and so $(1-x)$ is a unit. Now given a max right ideal I , if $x \notin I$ then $xR + I = R$, so $xw + y = 1$ some $w \in R$, $y \in I$. $xw \in J_1$ (we did prove that J_1 and J_2 were ideals) so the same argument as above shows $1 - xw$ is a unit, a contradiction. Conclude that $x \in J_2$; by symmetry the reverse inclusion holds.

6. (a) Show that a one-sided ideal I in a ring R is contained in $J(R)$ if and only if $1 - x$ is a unit for all $x \in I$.

(b). The left ideal I is *nil* if every element of I nilpotent. Show that any nil left ideal is contained in $J(R)$. (Hint: if $x^n = 0$ then show $(1 - x)$ is a unit.)

3 The Weddeburn theorem and semisimple rings and modules

7. Prove that if M is a semisimple module, then all submodules and factor modules of M are semisimple. (This is argued in the text in a slightly different way; so do this only if you have not read the argument in the text.)

(Sketch. write $M = \bigoplus_{\alpha \in A} S_{\alpha}$ for some index set A , where each module S_{α} is simple. Let N be a submodule of M . Show using Zorn's lemma that there is a subset $B \subseteq A$ which is maximal with respect to the property that $N \cap N' = 0$, where $N' = \bigoplus_{\alpha \in B} S_{\alpha} = 0$. Prove that $N + N' = M$ and conclude that $M \cong N \oplus N'$. Then $M/N \cong N'$ is semisimple and so all factor modules of M are semisimple. Then $N \cong M/N'$ is also semisimple.)

8. Let R be a ring and $M = S_1^{\oplus n_1} \oplus \cdots \oplus S_d^{\oplus n_d}$ be a direct sum where each S_i is a simple left R -module and $S_i \not\cong S_j$ for $i \neq j$.

Show that $\text{End}_R(M) \cong M_{n_1}(D_1) \times \cdots \times M_{n_d}(D_d)$ as rings, where $D_i = \text{End}_R(S_i)$, where each D_i is a division ring. (you may quote Schur's lemma.)

9. Let K be an algebraically closed field, and suppose that D is a division ring which is a K -algebra, such that $\dim_K D < \infty$. Show that $D = K$. (Hint: given $a \in D$, $\sum_{i=0}^{\infty} K a^i$ is a subfield of D .)

10. Let $R = K[x][y; \sigma]$ be the quantum plane, where $\sigma : x \rightarrow qx$ for some q which is a primitive n th root of 1 in K . Assume that the field K is algebraically closed.

(a). Following the method we used in class in the non-root of one case, show that if I is a prime ideal of R which contains a nonzero element of the form $\sum_{0 \leq i, j \leq n-1} a_{ij} x^i y^j$ where $a_{ij} \in K$, then I contains either x or y .

(b). Let $J = R(x^n - a, y^n - b)$, where $a, b \neq 0$. Show that this is a two sided ideal (note x^n, y^n are central) and that R/J is a ring of vector space dimension exactly n^2 over K (the elements $\{x^i y^j | 0 \leq i, j \leq n-1\}$ form a K -basis.)

(c). Prove that J is a maximal ideal in R , and that $R/J \cong M_n(K)$ as rings. (Hint: for the second statement, apply Wedderburn and use the previous exercise.)

4 GK-dimension

11. Recall that for a finitely generated K -algebra R , the GK-dimension of R is defined by taking any finite-dimensional K -subspace V of R which contains 1 and generates R as an

algebra, and setting

$$\mathrm{GK}(R) = \limsup(\log_n \dim_K V^n).$$

Suppose that R is an algebra with generating subspace V as above, and that $\alpha : R \rightarrow R$ is an automorphism of R and $\delta : R \rightarrow R$ is a α -derivation, such that $\alpha(V) \subseteq V$ and $\delta(V) \subseteq V$. Suppose that $\mathrm{GK}(R)$ is finite. Show that $\mathrm{GK}(R[x; \alpha, \delta]) = \mathrm{GK}(R) + 1$.

12. Show that $\mathrm{GK}(A_n) = 2n$, where A_n is the n th Weyl algebra over the field K . Similarly, show that if $R = K[x][y; \sigma]$ where $\sigma : x \rightarrow qx$ for some $0 \neq q \in K$ (i.e. R is the quantum plane) then $\mathrm{GK}(K[x][y; \sigma]) = 2$.