Math 207A: Hopf Algebras

(Lecture Notes)

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Contents

These notes are (student) typed up lecture notes for the UCSD Math 207A class in Spring 2020 covering Hopf Algebras.

DK Note: Maybe write a bit more later.

Chapter 1

Algebras and Coalgebras

cha:coalgs

1.1 Algebra basics

Let k be a field.

Definition 1.1 (k-algebra). A k-algebra A is a ring with 1 which is also a k-vector space, such that

$$\lambda \cdot (ab) = (\lambda \cdot a)b = a(\lambda \cdot b)$$

for all $\lambda \in k$ and $a, b \in A$.

Given a k-algebra A, there is a ring homomorphism

$$\begin{aligned} u\colon k &\to A \\ \lambda &\mapsto \lambda \cdot 1 \end{aligned}$$

with $u(k) \subseteq Z(A)$ (the center of A).

Conversely, given a ring A and such a homomorphism $u\colon k \to A$ such that $u(k) \subseteq Z(A)$, we have that A is a k-algebra with scalar multiplication $\lambda \cdot a = u(\lambda)a$. Moreover, since we are only concerned with algebras over a field in these notes, the homomorphism u must be injective, so we can identify k with $u(k)$ and think of $k \subseteq A$ (where $k \subseteq Z(A)$).

Example 1.2 (Monoid algebra). Let M be a monoid., i.e. M is a set with a binary operation (=a product) which is associative and has a unit. Then we can define the *monoid algebra* kM, which is a k-vector space with basis M and product induced by extending the product of M linearly. More formally, we may write an arbitrary element of kM as $\sum_m a_m m$, where if M is infinite then all but finitely many a_m are 0.

So

$$\Big(\sum_{m \in M} a_m m\Big)\Big(\sum_{n \in M} b_n n\Big) = \sum_{m \in M}\sum_{n \in M} a_m b_n mn,$$

where $a_m = 0$ for all but finitely many $n \in M$, $b_n = 0$ for all but finitely many $n \in M$, and mn is the product in the monoid M. Note that the identity element of kM is the identity element 1_M of the monoid M. ○

A group is just a monoid for which every element has a multiplicative inverse. We will be especially interested in the special case of group algebras below.

Example 1.3. As a special case of the preceding example, consider the monoid $(\mathbb{N}, +)$ written multiplicatively as $\mathbb{N} = \{x^0, x^1, x^2, \ldots\}$, so that $x^i x^j = x^{i+j}$. Then the monoid algebra $k\mathbb{N}$ is isomorphic to the algebra of polynomials $k[x]$.

If we instead take the monoid $(\mathbb{Z}, +)$ written multiplicatively as $\mathbb{Z} = \{\ldots, x^{-2}, x^{-1}, x^0, x^1, x^2, \ldots\}$, then we get the algebra of Laurent polynomials, i.e. $k\mathbb{Z} \cong k[x, x^{-1}]$, and we call it a group algebra. ○

One should be familiar with many more examples of algebras from the study of ring and module theory. In this course many of the main results will concern finite dimensional algebras, that is k-algebras A for which $\dim_k A < \infty$. If M is a finite monoid then kM is such an example. Another simple example is the ring $M_n(k)$ of $n \times n$-matrices with entries in k.

We will make heavy use of tensor products in this course, but primarily tensor products over a field (k), which are especially easy to understand. We won't review the general definition and theory of tensor products here. Recall, however, that if V and W are k-vector spaces, with respective k-bases $\{v_i \mid i \in I\}$ and $\{w_j \mid j \in J\}$, then the tensor product $V \otimes_k W$ has a k-basis of pure tensors $\{v_i \otimes w_j \mid (i,j) \in I \times J\}$. This gives a very explicit way of thinking of a tensor product over a field, though for some purposes it is better to rely on the universal property of the tensor product rather than thinking in terms of bases.

Universal Property (Tensor product of k-vector spaces): *For any bilinear, k-balanced map $\phi\colon M \times N \to L$, there exists a unique k-linear map $\widehat{\phi}\colon M \otimes N \to L$ such that $\phi(m,n) = \widehat{\phi}(m \otimes n)$. Equivalently, $\phi = \widehat{\phi} \circ f$, where $f\colon M \times N \to M \otimes N$ is given by $f(m,n) = m \otimes n$, i.e. we get the commutative diagram:*

$$\begin{tikzcd}
 & M \otimes N \arrow[d, dashed, "\exists!\widehat{\phi}"] \\
M \times N \arrow[ur, "f"] \arrow[r, "\phi"'] & L
\end{tikzcd}$$

We refer to bilinear and balanced over k as "k-bilinear" from now on.

ex:sumandtenalg **Example 1.4 ($\oplus$ and $\otimes$).** Let A and B be k-algebras. Then the direct sum $A \oplus B$ (as vector spaces) is naturally a k-algebra, with product

$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2)$$

and scalar product

$$\lambda \cdot (a, b) = (\lambda a, \lambda b).$$

The tensor product $A \otimes_k B$ is also a k-algebra with product induced by the extending linearly the product on pure tensors given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2),$$

and with scalar multiplication

$$\lambda \cdot (a_1 \otimes b_1) = (\lambda a_1) \otimes b_1 = a_1 \otimes (\lambda b_1).$$ ○

1.2 Diagrammatic definition of an algebra

We would like to "dualize" the definition of an algebra. In order to do this we need to first express the definition of algebra in terms of commutative diagrams.

First, note that if A is a k-algebra, then the map

$$\begin{aligned} A \times A &\to A \\ (a, b) &\mapsto ab \end{aligned}$$

is k-bilinear. Thus by the universal property of the tensor product, we get a unique k-linear map

$$\begin{aligned} A \otimes_k A &\to A \\ a \otimes b &\mapsto ab, \end{aligned}$$

which we refer to as the *multiplication map* of the algebra. As noted earlier, we can also think of the k-vector space structure in terms of a ring homomorphism

$$\begin{aligned} u \colon k &\to A \\ \lambda &\mapsto \lambda \cdot 1 \end{aligned}$$

with $u(k) \subseteq Z(A)$, which we refer to as *unit map*.

In the following result, we use that there are canonical isomorphisms $k \otimes_k V \cong V$ and $V \otimes_k k \cong V$ for any k-vector space V, and take these a identifications. From now on, because almost all tensor products will be over the field k, we write $\otimes_k$ as $\otimes$ when there is no chance of confusion. We write id_S for the identity map $S \to S$ of any set S, or sometimes just id if the set is clear.

`lem:algstruct` **Lemma 1.5.** *Suppose that A is a k-vector space together with k-linear maps $m\colon A \otimes A \to k$ and $u\colon k \to A$. These maps give A the structure of a k-algebra for which m and u are the multiplication and unit maps, if and only if the following two diagrams are commutative:*

$$
\begin{tikzcd}
A \otimes_k k \arrow[r, equal] \arrow[d, "\mathrm{id}_A \otimes u"'] & A \arrow[r, equal] \arrow[d, "\mathrm{id}_A"] & k \otimes_k A \arrow[d, "u \otimes \mathrm{id}_A"] \\
A \otimes A \arrow[r, "m"'] & A & A \otimes A \arrow[l, "m"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}_A"] \arrow[d, "\mathrm{id}_A \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"'] & A
\end{tikzcd}
$$

Proof. The commutativity of the first diagram says that

$$
\begin{tikzcd}
a \otimes 1_k \arrow[r, equal] \arrow[d, mapsto] & a \arrow[r, equal] \arrow[d, equal] & 1_k \otimes a \arrow[d, mapsto] \\
a \otimes u(1_k) \arrow[r, mapsto] & au(1_k) = u(1_k)a & u(1_k) \otimes a, \arrow[l, mapsto]
\end{tikzcd}
$$

and thus $u(1_k) = 1_A$ is the identity element of A. And the commutativity of the second diagram says that

$$
\begin{tikzcd}
a \otimes b \otimes c \arrow[r, mapsto] \arrow[d, mapsto] & ab \otimes c \arrow[d, mapsto] \\
a \otimes bc \arrow[r, mapsto] & a(bc) = (ab)c,
\end{tikzcd}
$$

i.e. the product given by m is associative.

The left diagram also says that for $\lambda \in k, a \in A$, $\lambda \cdot a = (\lambda \cdot 1_A)a = a(\lambda \cdot 1)$, so $k1_A = u(k)$ is in the center of A. $\square$

We sometimes refer to an algebra by the triple (A, m, u) of the k-vector space A and the two maps m and u that define the algebra structure.

1.3 Coalgebras

The definition of a coalgebra is made by reversing the arrows in the diagrams in Lemma 1.5. This leads to a notion that seems much less intuitive than an algebra at first, but we will see that there are many examples.

`def:coalg` **Definition 1.6 (Coalgebra).** Suppose that C is a k-vector space together with k-linear maps $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$. Then C is called a *coalgebra*, and the maps Δ and ε are called the *comultiplication* (or *coproduct*) and the *counit* respectively, if the following two diagrams are commutative:

$$
\begin{tikzcd}
C \otimes C \arrow[d, "\mathrm{id}_C \otimes \varepsilon"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}_C"] & C \otimes C \arrow[d, "\varepsilon \otimes \mathrm{id}_C"] \\
C \otimes_k k \arrow[r, equal] & C \arrow[r, equal] & k \otimes_k C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}_C"] \\
C \otimes C \arrow[r, "\mathrm{id}_C \otimes \Delta"'] & C \otimes C \otimes C
\end{tikzcd}
$$

Remark. The second diagram is called "coassociativity" of Δ. $\triangle$

We sometimes refer to a coalgebra by the triple (C, Δ, ε) of the k-vector space C and the two maps Δ and ε that define the coalgebra structure.

Many common examples of product operations defining algebras involve combining two elements in a natural way such as multiplication of numbers or composition of functions. Conversely, many natural coproducts take an element and pull it apart into two pieces in all possible ways, summing over the possibilities.

ex:monoidcoalg **Example 1.7 (Monoid coalgebra).** Let M be a monoid with the property that for all $m \in M$, there are finitely many pairs $(n, p) \in M \times M$ such that $np = m$. Let kM be the vector space with basis given by the elements of M, and define a coalgebra structure on kM with

$$\Delta(m) = \sum_{(n,p)\in\{(n,p)\in M^2 | np=m\}} n \otimes p \quad \forall m \in M,$$
$$\varepsilon(m) = \begin{cases} 0 & \text{if } m \neq 1_M \\ 1 & \text{if } m = 1_M \end{cases}$$

for $m \in M$ extended linearly to all of kM.

We claim that $(kM, \Delta, \varepsilon)$ is a coalgebra, which we refer to as the *monoid coalgebra of* M. Since both Δ and ε are defined on the object of M and then extended linearly to kM, it is easy to see that to check the necessary diagrams hold it is enough to check they commute when starting with an element $m \in M$. This is because all of the maps in the diagrams are k-linear.

For the top diagram we note that

$$(\mathrm{id} \otimes \varepsilon)(\Delta(m)) = (\mathrm{id} \otimes \varepsilon)\Big(\sum_{np=m} n \otimes p\Big) = \sum_{np=m} \varepsilon(n)p = m.$$

This is clear since every summand is zero except for the one with $n = 1_M$, $p = m$. Similarly

$$(\varepsilon \otimes \mathrm{id})(\Delta(m)) = \sum_{np=m} n\varepsilon(p) = m,$$

so the left diagram commutes. Next, we note that

$$(\Delta\otimes\mathrm{id})\circ\Delta(m) = (\Delta\otimes\mathrm{id})\Big(\sum_{np=m} n\otimes p\Big) = \sum_{qr=n}\sum_{np=m} q\otimes r\otimes p = \sum_{qrp=m} q\otimes r\otimes p,$$

and similarly

$$(\mathrm{id}\otimes\Delta)\circ\Delta(m) = (\mathrm{id}\otimes\Delta)\Big(\sum_{np=m} n\otimes p\Big) = \sum_{st=p}\sum_{np=m} n\otimes s\otimes t = \sum_{nst=m} n\otimes s\otimes t,$$

so we see that these are the same, and thus the right diagram also commutes. $\bigcirc$

Example 1.8. As a special case of the previous example, consider the multiplicative monoid $\mathbb{N} = \{x^0, x^1, x^2, \ldots\}$. The colagebra $k\mathbb{N}$ defined above has coproduct and counit given by

$$\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j,$$
$$\varepsilon(x^n) = \delta_{0n},$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

is the "Kronecker delta". ○

Example 1.9 (Trivial coalgebra). The field k is a coalgebra in a canonical way, where $\Delta\colon k \to k \otimes k = k$ and ε are both the identity maps under the natural identification of $k \otimes_k k$ with k. This is called the *trivial coalgebra*. ○

Example 1.10 (Matrix coalgebra). Let $A = M_n(k)$ be the k-algebra of $n \times n$-matrices with entries in k. For $1 \leq i, j \leq n$ let e_{ij} be the matrix with a 1 in the (i, j)-entry and 0 in all other entries. Then it is easy to check that $e_{ij}e_{st} = \delta_{js}e_{it}$. The elements $\{e_{ij} \mid 1 \leq i, j \leq n\}$ are traditionally called *matrix units* and they clearly form a basis for A.

Now we can give A a coalgebra structure by defining

$$\Delta(e_{ij}) = \sum_{1 \leq s \leq n} e_{is} \otimes e_{sj}$$

and

$$\varepsilon(e_{ij}) = \delta_{ij},$$

and extend linearly to A. Note that

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(e_{ij}) = (\varepsilon \otimes \mathrm{id})\Big(\sum_s e_{is} \otimes e_{sj}\Big) = \sum_s \varepsilon(e_{is})e_{sj} = \varepsilon(e_{ii})e_{ij} = e_{ij},$$

and similarly

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta(e_{ij}) = (\mathrm{id} \otimes \varepsilon)\Big(\sum_s e_{is} \otimes e_{sj}\Big) = \sum_s e_{is}\varepsilon(e_{sj}) = e_{ij}\varepsilon(e_{jj}) = e_{ij},$$

so the left diagram in Definition 1.6 commutes. Similar checks for the right diagram shows that (A, Δ, ε) is a coalgebra. ○

Example 1.11 (Grouplike coalgebra on S). Let S be any set and let kS be a vector space space with basis S. Define

$$\begin{aligned}\Delta\colon kS &\to kS \otimes kS\\ s &\mapsto s \otimes s,\\ \varepsilon\colon kS &\to k\\ s &\mapsto 1\end{aligned}$$

for all $s \in S$. Extending Δ and ε linearly, we get a coalgebra kS. We call the coalgebra $(kS, \Delta, \varepsilon)$ the *grouplike coalgebra on S*

More generally, in any coalgebra (C, Δ, ε), an element $c \in C$ with $\Delta(c) = c \otimes c$ and $\varepsilon(c) = 1$ is called a *grouplike element.* Thus the grouplike coalgebra has a k-basis of grouplike elements. ○

ex:sumandtenofcoalgs **Example 1.12 ($\oplus$ and $\otimes$).** Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras over k. Then $C \oplus D$ is naturally a coalgebra, as follows. Note that we have a canonical isomorphism

$$(C \oplus D) \otimes (C \oplus D) \cong (C \otimes C) \oplus (D \otimes C) \oplus (C \otimes D) \oplus (D \otimes D).$$

Then we define the coproduct Δ of $C \oplus D$ as the composition

$$\begin{tikzcd} C \oplus D \arrow[r, "\Delta_C \oplus \Delta_D"] \arrow[dr, "\Delta"'] & (C \otimes C) \oplus (D \otimes D) \arrow[d, "\iota"] \\ & (C \oplus D) \otimes (C \oplus D), \end{tikzcd}$$

where ι is the natural inclusion we get by the above isomorphism. Defining also

$$\varepsilon((c, d)) = \varepsilon_C(c) + \varepsilon_D(d),$$

it is straightforward to check that $(C \oplus D, \Delta, \varepsilon)$ is a coalgebra. We can also describe the coproduct of $C \oplus D$ more explicitly, as follows. For $c \in C$ we write

$$\Delta_C(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2},$$

and similarly for $d \in D$ we write

$$\Delta_D(d) = \sum_{j=1}^{n} d_{j,1} \otimes d_{j,2}.$$

Then

$$\Delta((c, d)) = \sum_{i=1}^{m} (c_{i,1}, 0) \otimes (c_{i,2}, 0) + \sum_{j=1}^{n} (0, d_{j,1}) \otimes (0, d_{j,2}).$$

The tensor product $C \otimes D$ also has a coalgebra structure. The coproduct Δ is the composition

$$\begin{tikzcd} C \otimes D \arrow[r, "\Delta_C \otimes \Delta_D"] \arrow[dr, "\Delta"'] & (C \otimes C) \otimes (D \otimes D) \arrow[d, "\tau_{23}"] \\ & (C \otimes D) \otimes (C \otimes D), \end{tikzcd}$$

where

$$\tau_{23}(c_1 \otimes c_2 \otimes d_1 \otimes d_2) = c_1 \otimes d_1 \otimes c_2 \otimes d_2$$

is the function that switches the second and third tensorands. The counit is given by

$$\varepsilon(c \otimes d) = \varepsilon(c)\varepsilon(d).$$

Again, if we wish, we can write the formula for the coproduct Δ explicitly on elements as

$$\Delta(c \otimes d) = \sum_{i=1}^{m} \sum_{j=1}^{n} (c_{i,1} \otimes d_{j,1}) \otimes (c_{i,2} \otimes d_{j,2}). \qquad \bigcirc$$

1.4 Sweedler Notation

If we need to write the action of a coproduct on an element of a colagebra C in coordinates, formally we get something like the notation $\Delta(c) = \sum_{i=1}^{m} c_{i,1} \otimes c_{i,2}$ we wrote in the previous example. This is awkward in several ways: the number of summands m depends on the element c in general; the double indexing is ugly; and if we had to apply $\Delta \otimes \mathrm{id}_C$ to $\Delta(c)$ even more indices would appear.

Moss Sweedler invented a simplified notation which is in wide use. One just writes $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. The number of summands is undetermined, and one does not index the elements at all, other than indicating their positions in the tensor product. This may be confusing at first, but once one gets used to this notation it adds great clarity to proofs in which one needs to work with the action of the coproduct on an arbitrary element.

By the axioms of a coalgebra, $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$. Sometimes this map is written as $\Delta^{(2)} \colon C \to C \otimes C \otimes C$. In Sweedler notation, applying the first operation to c gives

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta(c) = (\Delta \otimes \mathrm{id}_C)\Big(\sum c_{(1)} \otimes c_{(2)}\Big) = \sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}.$$

On the other hand,

$$(\mathrm{id}_C \otimes \Delta) \circ \Delta(c) = (\mathrm{id}_C \otimes \Delta)\Big(\sum c_{(1)} \otimes c_{(2)}\Big) = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}.$$

Now the expressions $\sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$ and $\sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$ must represent the same element of $C \otimes C \otimes C$. Sweedler's notation simplifies this further, and represents this element as

$$\Delta^{(2)}(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.$$

In this way, double Sweedler indices can be avoided. Notice that the indices just refer to the positions of the tensorands and not specific elements.

In Sweedler notation, the axiom of the counit ε takes the following form: for all $c \in C$, one has

$$c = \sum \varepsilon(c_{(1)})c_{(2)} = \sum c_{(1)}\varepsilon(c_{(2)}).$$

There is an alternative notation which simplifies even further by avoiding the parentheses and just writes $\Delta(c) = \sum c_1 \otimes c_1$. If one is feeling especially lazy, one even omits the sum and writes $\Delta(c) = c_{(1)} \otimes c_{(2)}$. This is a bit dangerous since one must constantly remember that there is an implicit sum and $c_{(1)} \otimes c_{(2)}$ does not stand for a pure tensor.

We will begin to use the Sweedler notation and we will soon see how it works in practice.

1.5 Basic properties of vector space duals

In this section we give reminders about the basic properties of vector space duals. Let V be a vector space over the field k. The *dual space* is

$$\begin{aligned} V^* &= \mathrm{Hom}_k(V, k) \\ &= \{\text{all } k\text{-linear maps } V \text{ to } k\} \\ &= \{\text{linear functionals on } V\}. \end{aligned}$$

that is, the collection of all linear transformations $f\colon V \to k$. Such an f is sometimes called a *linear functional.* V^* is naturally itself a vector space with pointwise operations: if $f, g \in V^*$ and $\lambda \in k$, then $f + g \in V^*$ and $\lambda f \in V^*$ where

$$\begin{aligned} [f+g](v) &= f(v) + g(v), \\ [\lambda f](v) &= \lambda f(v) \end{aligned}$$

for $v \in V$.

Suppose that $\dim_k V < \infty$ and $v_1, \dots, v_n$ is a k-basis of V. Then we define the *dual basis* of V^* to be $\{v_1^*, \dots, v_n^*\}$, where

$$v_i^*(v_j) = \delta_{ij}.$$

It is easy to check that $\{v_1^*, \dots, v_n^*\}$ is a basis for V^*; in particular, $\dim_k V^* = n = \dim_k V$. Since the vector spaces V and V^* have the same dimension, there is a vector space isomorphism between them, but there is no *canonical* vector space isomorphism $V \to V^*$. In particular the isomorphism

$$\begin{aligned} V &\to V^* \\ v_i &\mapsto v_i^* \end{aligned}$$

is highly dependent on the basis.

If $\dim_k V = \infty$, then V^* and V cannot be isomorphic as vector spaces, as the dimension of the vector space V^* has a larger cardinality than the dimension of V, though we don't prove that here.

We write V^{**} for the double dual $(V^*)^* = \mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k)$. There is a canonical linear transformation

$$\begin{aligned} i\colon V &\to V^{**} \\ v &\mapsto e_v, \end{aligned}$$

where

$$\begin{aligned} e_v\colon V^* &\to k \\ f &\mapsto f(v) \end{aligned}$$

is "valuation at v". The map i is always injective. If V is finite dimensional over k, then we have $\dim_k V = \dim_k V^* = \dim_k V^{**}$ as observed above. Thus in this case i is an isomorphism since it is an injective linear transformation between vector spaces of the same finite dimension. We see that V and V^{**} are *canonically* isomorphic in this case. If V is infinite dimensional over k, then i cannot be an isomorphism, since it is again the dimension of V^{**} of a cardinality larger than the dimension of V.

Suppose $\phi\colon V \to W$ is a linear transformation of k-vector spaces. Then there is an induced "dual" linear transformation of the dual spaces,

$$\begin{aligned} \phi^*\colon W^* &\to V^* \\ f &\mapsto f \circ \phi \end{aligned} \tag{1.1}$$ {eq:pullback}

called the *pullback by* ϕ.

If $i_V\colon V \to V^{**}$ is the canonical map given above, then there is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle i_V}\downarrow & & \downarrow{\scriptstyle i_W} \\ V^{**} & \xrightarrow[\phi^{**}]{} & W^{**}. \end{array}$$

If V and W are finite dimensional, then i_V and i_W are isomorphisms, so the diagram says that dualizing twice, the map ϕ^{**} is essentially the same as ϕ, i.e. we can identify ϕ^{**} with ϕ.

Duals and $\otimes$

It is useful also to notice how duals interact with the tensor product. For vector spaces V and W we always have a canonical linear transformation $\iota\colon V^* \otimes W^* \to (V \otimes W)^*$, where

$$[\iota(f \otimes g)](v \otimes w) = f(v)g(w).$$

One may check that ι is always an injective linear transformation, and that when either V or W is finite dimensional, then ι is an isomorphism, but not when both V and W are infinite dimensional. This can be proved by choosing bases.

1.6 Duals of Algebras and Coalgebras

Let C be a coalgebra over k. We claim that the dual space C^* has a natural k-algebra structure.

For $f, g \in C^* = \mathrm{Hom}_k(C, k)$, we define $fg \in C^*$ by

$$[fg](c) = \sum f(c_{(i)}) \otimes g(c_{(2)}) = (f \otimes g) \circ \Delta(c),$$

where

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$$

in Sweedler notation. Note that the counit of C is a linear map $\varepsilon\colon C \to k$, so $\varepsilon \in C^*$. We claim that the product above is an associative product on C^* and that ε is the unit element.

We give a direct proof that C^* is an algebra. By its definition, it is clear that the relation sending $(f, g) \mapsto [fg]$ is k-bilinear, and so induces a linear map $m\colon C^* \otimes C^* \to C^*$. To check that m defines an associative product, we calculate for $f, g, h \in C^*$ that

$$\begin{aligned}[(fg)h](c) &= \sum (fg)(c_{(1)})h(c_{(2)}) = \sum f(c_{(1)(1)})g(c_{(1)(2)})h(c_{(2)}) \\ &= \sum f(c_{(1)})g(c_{(2)})h(c_{(3)})\end{aligned}$$

and likewise

$$\begin{aligned}[f(gh)](c) &= \sum f(c_{(1)})(gh)(c_{(2)}) = \sum f(c_{(1)})g(c_{(2)(1)})h(c_{(2)(2)}) \\ &= \sum f(c_{(1)})g(c_{(2)})h(c_{(3)}),\end{aligned}$$

where we have used the Sweedler notation for $\Delta^{(2)}$. To check that ε is then a unit for C^* we note that for all $c \in C$ and $f \in C^*$,

$$[\varepsilon f](c) = \sum \varepsilon(c_{(1)})f(c_{(2)}) = f\Big(\sum \varepsilon(c_{(1)})c_{(2)}\Big) = f(c),$$

where we have used that f is linear. Thus $\varepsilon f = f$. A similar argument shows that $f\varepsilon = f$. This shows that ε is the unit element, and so, more formally, the map $u\colon k \to C^*$ given by $u(\lambda) = \lambda\varepsilon$ is the unit map.

prop:dualofcoalg **Proposition 1.13.** *Let (C, Δ, ε) be a coalgebra. Define (A, m, u), $A = C^*$, $m = \Delta^* \circ i$ where $i \colon C^* \otimes C^* \to (C \otimes C)^*$ is the natural map, and $u = \varepsilon^*$, identifying k^* with k. Then (A, m, u) is an algebra.*

Proof. We have coassociativity, so the diagram

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta \otimes \mathrm{id}"] \\ C \otimes C \arrow[r, "\mathrm{id} \otimes \Delta"'] & C \otimes C \otimes C \end{tikzcd}$$

commutes. Dualizing, we get that

$$\begin{tikzcd} C^* & (C \otimes C)^* \arrow[l, "\Delta^*"'] \\ (C \otimes C)^* \arrow[u, "\Delta^*"] & (C \otimes C \otimes C)^* \arrow[l, "(\mathrm{id} \otimes \Delta)^*"] \arrow[u, "(\Delta \otimes \mathrm{id})^*"'] \end{tikzcd}$$

commutes. Expanding, we consider the diagram

$$\begin{tikzcd} C^* & (C \otimes C)^* \arrow[l, "\Delta^*"'] & C^* \otimes C^* \arrow[l, "i"'] \\ (C \otimes C)^* \arrow[u, "\Delta^*"] \arrow[ur, phantom, "\alpha"] & (C \otimes C \otimes C)^* \arrow[l, "(\mathrm{id}_C \otimes \Delta)^*"] \arrow[u, "(\Delta \otimes \mathrm{id}_C)^*"'] \arrow[ur, phantom, "\gamma"] & \\ C^* \otimes C^* \arrow[u, "i"] \arrow[ur, phantom, "\beta"] & & C^* \otimes C^* \otimes C^*. \arrow[ll, "\mathrm{id}_{C^*} \otimes (\Delta^* \circ i)"] \arrow[ul, "j"'] \arrow[uu, "(\Delta^* \circ i) \otimes \mathrm{id}_{C^*}"'] \end{tikzcd}$$

Here $j \colon C^* \otimes C^* \otimes C^* \to (C \otimes C \otimes C)^*$ is the canonical map for three vector spaces. We note that from above we have that square α commutes, and square β and γ can be shown to commute formally by an easy check. So the outside square commutes, and the outside map $C^* \otimes C^* \otimes C^* \to C^*$ says that $m = \Delta^* \circ i$ is associative; in other words the multiplication map diagram commutes.

Similarly, we dualize the counit diagram

$$\begin{tikzcd} C \otimes C \arrow[d, "\mathrm{id}_C \otimes \varepsilon"'] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}_C"] & C \otimes C \arrow[d, "\varepsilon \otimes \mathrm{id}_C"] \\ C \otimes k \arrow[r, equal] & C \arrow[r, equal] & k \otimes C, \end{tikzcd}$$

and consider the diagram

$$\begin{tikzcd} C^* \otimes C^* \arrow[r, "i"] & (C \otimes C)^* \arrow[r, "\Delta^*"] & C^* & (C \otimes C)^* \arrow[l, "\Delta^*"'] & C^* \otimes C^* \arrow[l, "i"'] \\ C^* \otimes k \arrow[u, "\mathrm{id}_{C^*} \otimes \varepsilon^*"] \arrow[r, equal] \arrow[ur, phantom, "\alpha'"] & (C \otimes k)^* \arrow[u, "(\mathrm{id}_C \otimes \varepsilon)^*"'] \arrow[r, equal] \arrow[ur, phantom, "\beta'"] & C^* \arrow[u, "\mathrm{id}_{C^*}"'] \arrow[r, equal] \arrow[ur, phantom, "\gamma'"] & (C \otimes k)^* \arrow[u, "(\varepsilon \otimes \mathrm{id}_C)^*"'] \arrow[r, equal] \arrow[ur, phantom, "\delta'"] & C^* \otimes k. \arrow[u, "\varepsilon^* \otimes \mathrm{id}_{C^*}"'] \end{tikzcd}$$

We note that the squares β' and γ' commute, because they are just the counit diagram dualized, and the squares α' and δ' can again be shown to commute formally by an easy check. Hence the left $\alpha'\beta'$ square and the right $\gamma'\delta'$ square commute, and, since $m = \Delta^* \circ i$ and $u = \varepsilon^*$, we see that the unit map diagram commutes. □

Remark. m in Proposition 1.13 is the same as the product on C^* defined above, i.e.

$$[(\Delta^* \circ i)(f \otimes g)](c) = \Delta^*(f \otimes g)(c) = (f \otimes g)(\Delta(c)) = \sum f(c_{(1)})g(c_{(2)})$$

for $f, g \in C^*$ and $c \in C$. △

Remark. If C is finite dimensional over k, then i is a canonical isomorphism, and we can identify $(C \otimes C)^*$ with $C^* \otimes C^*$. Then the duals of the coalgebra structure diagrams for C are exactly the algebra structure diagrams for $A = C^*$. △

Question: If A is an algebra, is A^* a coalgebra?

Note that $m^* \colon A^* \to (A \otimes A)^*$, so we would like to define $\Delta = \phi \circ m^*$, where $\phi \colon (A \otimes A)^* \to A^* \otimes A$ is natural. But such a ϕ does not exists in general, so:

Answer: In general, A (of arbitrary dimension) being an algebra does not imply that A^* is a coalgebra. ♣

Remark. If A is finite dimensional over k, then $i \colon A^* \otimes A^* \to (A \otimes A)^*$ is an isomorphism, so we can take $\phi = i^{-1}$ and this works. I.e. taking $A^* \otimes A^* = (A \otimes A)^*$ as an identification, A^* is a coalgebra. △

Corollary 1.14.

(1) Suppose that A is a finite dimensional k-algebra. Then A^{**} is also an algebra and $i_A \colon A \to A^{**}$ (the canonical map) is an isomorphism of algebras.

(2) Suppose that C is a finite dimensional coalgebra. Then C^{**} is also a coalgebra and $i_C \colon C \to C^{**}$ (the canonical map) is an isomorphism of coalgebras.

Example 1.15. Let S be a set, $(kS, \Delta, \varepsilon)$ the grouplike coalgebra with $\Delta(s) = s \otimes s$, $\varepsilon(s) = 1$ for $s \in S$. Let $A = (kS)^*$ be the dual algebra. Then, by definition,

$$A = \mathrm{Hom}_k(kS, k) = \mathrm{Hom}_{\mathbf{Sets}}(S, k),$$

so A is the set of functions $S \to k$ with pointwise operations. For example, if we take $f \colon S \to k$, $g \colon S \to k$, then

$$[fg](s) = (f \otimes g)(s \otimes s) = f(s)g(s).$$ ○

Example 1.16. Let $C = M_n(k)$ be the coalgebra with operations

$$\Delta(e_{ij}) = \sum_{\ell=1}^{n} e_{i\ell} \otimes e_{\ell j},$$
$$\varepsilon(e_{ij}) = \delta_{ij},$$

where the e_{ij}'s are the matrix units and δ_{ij} is the Kronecker delta. To describe $A = C^*$, consider the dual basis $\{e^*_{ij} \mid 1 \leq i, j \leq n\}$ (with $e^*_{ij}(e_{k\ell}) = \delta_{ik}\delta_{j\ell}$) and note that

$$[e^*_{ij}e^*_{\ell m}](e_{rs}) = \sum_t e^*_{ij}(e_{rt})e^*_{\ell m}(e_{ts}) = \sum_t \delta_{ir}\delta_{jt}\delta_{\ell t}\delta_{ms} = \delta_{ir}\delta_{j\ell}\delta_{ms},$$

so

$$e^*_{ij}e^*_{\ell m} = \delta_{j\ell}e^*_{im}.$$

This agrees with the usual multiplication of matrix units, i.e. $e_{ij}e_{\ell m} = \delta_{j\ell}e_{im}$, so it shows that $C^* \cong M_n(k)$ as an algebra, since the basis elements e^*_{ij} multiply exactly as matrix units in $M_n(k)$ do. Finally it is easy to see that $\varepsilon^* = e^*_{11} + \cdots + e^*_{nn}$ by noting that multiplying this on either side of any e^*_{ij} returns e^*_{ij}. ○

Example 1.17. Let M be a finite monoid and consider the monoid algebra $A = kM$, and let $C = A^*$ with the dual basis $\{p^* \mid p \in M\}$. Then C is a coalgebra with

$$\Delta(p^*) = \sum_{\substack{q,r \in M \\ qr = p}} q^* \otimes r^*,$$
$$\varepsilon(p^*) = \begin{cases} 0 & \text{if } p \neq 1_M, \\ 1 & \text{if } p = 1_M. \end{cases}$$

This is the monoid coalgebra described in Example 1.7.

Note that $\Delta(p^*) \in A^* \otimes A^*$ has the property that $\Delta(p^*)(q \otimes r) = p^*(qr)$ for $q, r \in M$ (since $\Delta = m^*$), which agrees with the above. ○

1.7 Coalgebra terminology

To continue our study of coalgebras we will need to introduce some standard (categorical) terminology. Specifically we need to introduce a description of homomorphisms and isomorphisms in the category of (k-)coalgebras, and we need to describe sub-objects, factor-objects and the like in the category of coalgebras.

def:coalgterm

Definition 1.18 (Coalgebra morphism, kernel, image and isomorphism). Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras.

A linear map $f\colon C \to D$ is a morphism of coalgebras if the following diagrams are commutative:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f \otimes f"'] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\varepsilon_C"'] & & D \arrow[dl, "\varepsilon_D"] \\ & k & \end{tikzcd}$$

Kernel and image of f are as usual for the linear map f.

f is an isomorphism if it is bijective.

Definition 1.19 (Coideal and subcoalgebra). Let C be a coalgebra.

A subspace $I \subseteq C$ is a *coideal* if

$$\Delta(I) \subseteq I \otimes C + C \otimes I$$

and

$$\varepsilon(I) = 0.$$

A subspace $D \subseteq C$ is a *subcoalgebra* of C if

$$\Delta(D) \subseteq D \otimes D.$$

lem:kernelpi

Lemma 1.20. *Let V and W be vector spaces with subspaces $V' \subseteq V$ and $W' \subseteq W$. Consider the vector space map*

$$\begin{aligned} \pi\colon V \otimes W &\to V/V' \otimes W/W' \\ v \otimes w &\mapsto (v + V') \otimes (w + W'). \end{aligned}$$

Then

$$\ker \pi = V' \otimes W + V \otimes W'.$$

Proof. It is clear that $V' \otimes W + V \otimes W' \subseteq \ker \pi$.

Now choose vector space complements (e.g. by picking a basis) $V'' \subseteq V$ such that $V = V' \oplus V''$ and $W'' \subseteq W$ such that $W = W' \oplus W''$. Then $\pi|_{V'' \oplus W''}$ is an isomorphism onto $V/V' \otimes W/W'$, and thus

$$V'' \oplus W'' \cap \ker \pi = 0.$$

But, we also have that

$$(V'' \otimes W'') \oplus (V' \otimes W + V \otimes W') = V \otimes W,$$

so

$$\ker \pi = \ker \pi \cap V \otimes W \subseteq V' \otimes W + V \otimes W',$$

and thus

$$\ker \pi = V' \otimes W + V \otimes W'. \qquad \square$$

Proposition 1.21. *Let C and D be coalgebras, and let $f: C \to D$ be a morphism of coalgebras.*

(1) If V is a subcoalgebra of C, then V is a coalgebra with $\Delta_V = \Delta_C|_V$ and $\varepsilon_V = \varepsilon_C|_V$.

(2) If I is a coideal of C, then C/I is a factor coalgebra with

$$\Delta_{C/I}(c+I) = \sum (c_{(1)} + I) \otimes (c_{(2)} + I),$$
$$\varepsilon_{C/I}(c+I) = \varepsilon_C(c).$$

(3) $\operatorname{Ker} f$ is a coideal of C and $\operatorname{Im} f = f(C)$ is a subcoalgebra of D.

(4)

$$\widetilde{f}: C/\operatorname{Ker} f \to f(C)$$
$$c + \operatorname{Ker} f \mapsto f(c)$$

is an isomorphism of coalgebras.

Proof. The details are left as an exercise to the reader. Note for (2), to show that $\Delta_{C/I}$ is well defined, we need to show that $\Delta_{C/I}(c) = 0$ for $c \in I$. But

$$\Delta(I) \subseteq I \otimes C + C \otimes I = \operatorname{Ker}(C \otimes C \to C/I \otimes C/I),$$

so $\Delta_{C/I}(c) = 0$. □

Example 1.22. Let $f: S \to T$ be a set map. Then f induces a linear map $\widetilde{f}: kS \to kT$, and one can check that $\widetilde{f}$ is in fact a morphism of grouplike coalgebras. We note that

$$\operatorname{Ker} \widetilde{f} = k\text{-}\operatorname{span}\{s_1 - s_2 \mid s_1, s_2 \in S \text{ with } f(s_1) = f(s_2)\}$$

and

$$\operatorname{Im} \widetilde{f} = kf(S).$$ ○

ex:strlowtrimat **Example 1.23.** Let $C = M_n(k)$ be the matrix coalgebra. Recall that

$$\Delta(e_{ij}) = \sum_t e_{it} \otimes e_{tj},$$
$$\varepsilon(e_{ij}) = \delta_{ij}.$$

Let

$$I = k\text{-}\operatorname{span}\{e_{ij} \mid i > j\} = \{\text{strictly lower triangular matrices}\},$$

and note that if $i > j$, then for all t we have either $t > j$ or $i > t$, so

$$\Delta(e_{ij}) \subseteq I \otimes C + C \otimes I$$

and $\varepsilon(e_{ij}) = 0$. Hence I is a coideal.

So we have a factor coalgebra

$$C/I = \{e_{ij} + I \mid i \leq j\}$$

and check that

$$\Delta(e_{ij} + I) = \sum_{i \leq t \leq j} (e_{it} + I) \otimes (e_{tj} + I)$$

and $\varepsilon(e_{ij} + I) = \delta_{ij}$. ○

1.8 Duality between substructures

Given a k-vector space V, there is a k-bilinear map

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle \colon V^* \times V &\to k \\ (f, v) &\mapsto f(v), \end{aligned}$$

which induces a linear map

$$\begin{aligned} V^* \otimes V &\to k \\ f \otimes v &\mapsto f(v). \end{aligned}$$

For $X \subseteq V$ a subset, we define

$$X^\perp = \{f \in V^* \mid \langle f, v\rangle = 0 \text{ for all } v \in X\},$$

and similarly for $Y \subseteq V^*$ a subset, we define

$$Y^\perp = \{v \in V \mid \langle g, v\rangle = 0 \text{ for all } g \in Y\}.$$

If V is a finite dimensional vector space, then we say that $\langle\,\cdot\,,\,\cdot\,\rangle$ is a *perfect pairing*, if for any *subspaces* $X \subseteq V$ and $Y \subseteq V^*$ we have that $X^{\perp\perp} = X$ and $Y^{\perp\perp} = Y$.

In the same way as above, we have a bilinear map

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle \colon (V^* \otimes V^*) \times (V \otimes V) &\to k \\ (f \otimes g, v \otimes w) &\mapsto f(v)g(w), \end{aligned}$$

and we define $\perp$ in as above in this setting as well. For $X \subseteq V \otimes V$ a subset, we define

$$X^\perp = \{f \in V^* \otimes V^* \mid \langle f, v\rangle = 0 \text{ for all } v \in X\},$$

and similarly for $Y \subseteq V^* \otimes V^*$ a subset, we define

$$Y^\perp = \{v \in V \otimes V \mid \langle g, v\rangle = 0 \text{ for all } g \in Y\}.$$

lem:IotimesJperp **Lemma 1.24.** *Let V be a vector space. If $I, J \subseteq V^*$ are subspaces, then*

$$(I \otimes J)^\perp = I^\perp \otimes V + V \otimes J^\perp.$$

Proof. Exercise. The argument is similar to one in the proof of Lemma 1.20. $\square$

Now we have:

thm:idealsubrels **Theorem 1.25.** *Let C be a coalgebra, and let $A = C^*$ be the dual algebra.*

1 If I is a ideal of $A = C^*$, then $I^\perp$ is a subcoalgebra of C.

2 If B is a subalgebra of $A = C^*$, then $B^\perp$ is a coideal of C.

3 If J is a coideal of C, then $J^\perp$ is a subalgebra of $A = C^*$.

4 If D is a subcoalgebra of C, then $D^\perp$ is an ideal of $A = C^*$.

Proof. Let I, J, K be subsapces of C^*.
Claim: $IJ \subseteq K$ implies that $\Delta(K^\perp) \subseteq I^\perp \otimes C + C \otimes J^\perp$.
Suppose that $IJ \subseteq K$ and note that the claim follows from Lemma 1.24 if

$$\Delta(K^\perp) \subseteq (I \otimes J)^\perp.$$

Now, for $c \in K^\perp \subseteq (IJ)^\perp$, we see for $f \in I$ and $g \in J$ that

$$0 = [fg](c) = (f \otimes g)(\Delta(c)),$$

so $\Delta(c) \in (I \otimes J)^\perp$, and the claim follows.

(1) If I is an ideal of C^*, then $C^*I \subseteq I$, so by the claim

$$\Delta(I^\perp) \subseteq \underbrace{(C^*)^\perp}_{=0} \otimes C + C \otimes I^\perp = C \otimes I^\perp.$$

Similarly $IC^* \subseteq I$, so by the claim

$$\Delta(I^\perp) \subseteq I^\perp \otimes C + C \otimes \underbrace{(C^*)^\perp}_{=0} = I^\perp \otimes C,$$

and thus

$$\Delta(I^\perp) \subseteq C \otimes I^\perp \cap I^\perp \otimes C = I^\perp \otimes I^\perp.$$

Hence $I^\perp$ is a subcoalgebra.

(2) If B is a subalgebra of C^*, then $BB \subseteq B$, so by the claim

$$\Delta(B^\perp) \subseteq B^\perp \otimes C + C \otimes B^\perp.$$

Also, since B is a subalgebra of C^*, we have that $\varepsilon = 1_{C^*} \in B$ and thus $\langle \varepsilon, c\rangle = 0$ for any $c \in B^\perp$, so

$$\varepsilon(B^\perp) = 0.$$

Hence $B^\perp$ is a coideal.

For (3) and (4), let U, V, W be subspaces of C and show that $\Delta(U) \subseteq V \otimes C + C \otimes W$ implies that $V^\perp W^\perp \subseteq U^\perp$. Then the proof can be finished by arguments as above. $\square$

Example 1.26. Consider the coalgebra $C = M_n(k)$ with

$$\Delta(e_{ij}) = \sum_t e_{it} \otimes e_{tj},$$
$$\varepsilon(e_{ij}) = \delta_{ij},$$

and thus $C^* \cong M_n(k)$ as an algebra with

$$e_{ij}^* e_{k\ell}^* = \delta_{jk} e_{i\ell}^*.$$

In Example 1.23 we saw that,

$$I = k\text{-}\operatorname{span}\{e_{ij} \mid i > j\} = \{\text{strictly lower triangular matrices}\}$$

is a coideal in C. So by Theorem 1.25,

$$I^\perp = \{e_{ij}^* \mid i \leq j\}$$

is a subalgebra of C^*. So $I^\perp$ is the subalgebra of upper triangular matrices in C^* (under the isomorphism $C^* \cong M_n(k)$).

Also the algebra C^* has no ideals except 0 and C^* since $M_n(k)$ is simple. So by Theorem 1.25, the only subcoalgebras of C are 0 and C. We say C is a *simple coalgebra.* $\bigcirc$

Corollary 1.27. *If C is a finite dimensional coalgebra, then there are bijections*

$$\{\textit{ideals of } C^*\} \longleftrightarrow \{\textit{subcoalgebras of } C\}$$

$$\{\textit{subalgebras of } C\} \longleftrightarrow \{\textit{coideals of } C\}$$

given by $(\,\cdot\,)^\perp$.

Now we get to one of our first results about coalgebras that doesn't really have a dual in the category of algebra.

Theorem 1.28. *Let C be a coalgebra and let $V \subseteq C$ be a finite dimensional subspace. Then $V \subseteq D \subseteq C$ for some subcoalgebra D with $\dim_k D < \infty$.*

Remark. The above theorem is sometimes called the fundamental theorem of coalgebras. $\triangle$

Corollary 1.29. *Let C be a coalgebra. Then*

$$C = \bigcup_{\substack{D \text{ finite dimensional} \\ \text{subcoalgebra of } C}} D.$$

Proof (of theorem). Fix a basis $\{c_i \mid i \in I\}$ for C, and write

$$\Delta(c_i) = \sum_{j,\ell} \alpha_{i,j,\ell} c_j \otimes c_\ell,$$

where $\alpha_{i,j,\ell} \in k$ and for a given i, $\alpha_{i,j,\ell} = 0$ except for finitely many (j, ℓ).

Let $\{v_1, \ldots, v_n\}$ be a basis for V, and write

$$\Delta(v_i) = \sum_j w_{i,j} \otimes c_j,$$

where $w_{i,j} \in C$ and for a given i, $w_{i,j} = 0$ except for finitely many j.

Let $W = \operatorname{span}\{v_i, w_{i,j}\} \subseteq C$ and consider $\Delta^{(2)}(v_i)$. We note that

$$\Delta^{(2)}(v_i) = \sum_j \Delta(w_{i,j}) \otimes c_j$$

and also

$$\Delta^{(2)}(v_i) = \sum_{s,t} \sum_j \alpha_{i,s,t} w_{i,j} \otimes c_s \otimes c_t,$$

so reindexing

$$\Delta^{(2)}(v_i) = \sum_{s,t,j} \alpha_{t,s,j} w_{i,t} \otimes c_s \otimes c_j.$$

So we get

$$\Delta(w_{i,j}) = \sum_{s,t} \alpha_{t,s,j} w_{i,t} \otimes c_s.$$

This shows that $\Delta(W) \subseteq W \otimes C$ since $\Delta(v_i) \subseteq W \otimes C$ and $\Delta(w_{i,j}) \subseteq W \otimes C$, i.e. W is a right coideal of C.

Now let $\{w_i\}$ be a basis for W and write (using the above)

$$\Delta(w_i) = \sum_j w_j \otimes b_{i,j},$$

where $b_{i,j} \in C$. Then

$$\Delta^{(2)}(w_i) = \sum_j \sum_\ell w_\ell \otimes b_{j,\ell} \otimes b_{i,j} = \sum_j w_j \otimes \Delta(b_{i,j}) = \sum_\ell w_\ell \otimes \Delta(b_{i,\ell}),$$

so

$$\Delta(b_{i,\ell}) = \sum_j b_{j,\ell} \otimes b_{i,j}.$$

Let $B = \operatorname{span}\{b_{i,j}\}$ and consider $D = B + W$. D is a subalgebra of C since

$$\Delta(W) \subseteq W \otimes B \subseteq D \otimes D,$$
$$\Delta(B) \subseteq B \otimes B \subseteq D \otimes D,$$

and $\dim_k D < \infty$. Finally, $V \subseteq W \subseteq D \subseteq C$. $\square$

This property has no dual for algebras in general. The dual property would say, if A is an algebra and $W \subseteq A$ a subspace such that $\dim_k A/W < \infty$, then there is an ideal $I \subseteq W \subseteq A$ with $\dim_k A/I < \infty$. But, this fails for general algebras, e.g. consider $A = k(x)$ the rational functions. Here A has no ideals I with A/I finite dimensional except $I = A$. (Note: Any simple infinite dimensional algebra would have the same problem.)

Chapter 2

Bialgebras

cha:bialgs

2.1 Bialgebras

Definition 2.1 (Bialgebra). A vector space B is a *bialgebra* if (B, m, u) is an algebra and (B, Δ, ε) is a coalgebra, and where either of the following equivalent properties holds:

(1) Δ, ε are algebra homomorphisms.

(2) m, u are coalgebra morphisms.

We usually refer to a bialgebra by the five tuple $(B, m, u, \Delta, \varepsilon)$.

ex:monbialg **Example 2.2.** Let M be a monoid, take (kM, m, u) to be the monoid algebra, and take $(kM, \Delta, \varepsilon)$ to be the grouplike coalgebra on M. Then $(kM, m, u, \Delta, \varepsilon)$ is a bialgebra. To see this, we will check (1) from the definition above: If $p, q \in M$, then

$$\Delta(pq) = pq \otimes pq = (p \otimes p)(q \otimes q) = \Delta(p)\Delta(q)$$

and

$$\Delta(1_M) = 1_M \otimes 1_M = 1_{kM \otimes kM}.$$

(Note that it is enough to check a product of basis elements to show that Δ is multiplicative.) So Δ is an algebra homomorphism. For ε we see that

$$\varepsilon(pq) = 1 = \varepsilon(p)\varepsilon(q),$$
$$\varepsilon(1_M) = 1. \qquad \bigcirc$$

Let's prove that the conditions (1) and (2) in the definition of a bialgebra are actually equivalent:

Proof. (1) says that

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \Delta(1) = 1,$$
$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \qquad \varepsilon(1) = 1,$$

so we get the following four diagrams.

$$\begin{tikzcd}
B \otimes B \arrow[r, "m_B"] \arrow[d, "\Delta_B \otimes \Delta_B"'] \arrow[dr, phantom, "\mathrm{I}"] & B \arrow[d, "\Delta_B"] \\
(B \otimes B) \otimes (B \otimes B) \arrow[r, "m_{B\otimes B}"'] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "u_B"] \arrow[dr, "u_{B\otimes B}"'] & B \arrow[d, "\Delta_B"] \arrow[dl, phantom, "\mathrm{II}"] \\
 & B \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
B \otimes B \arrow[r, "m_B"] \arrow[d, "\varepsilon_B \otimes \varepsilon_B"'] \arrow[dr, phantom, "\mathrm{III}"] & B \arrow[d, "\varepsilon_B"] \\
k \otimes k \arrow[r, equal] & k
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "u_B"] \arrow[dr, equal] & B \arrow[d, "\varepsilon_B"] \arrow[dl, phantom, "\mathrm{IV}"] \\
 & k
\end{tikzcd}$$

(2) says that m and u are coalgebra maps. $m_B \colon B \otimes B \to B$ being a coalgebra map means (cf. Definition 1.18) that

$$(m_B \otimes m_B) \circ \Delta_{B\otimes B} = \Delta_B \circ m_B,$$
$$\varepsilon_{B\otimes B} = \varepsilon_B \circ m_B,$$

which is equivalent to diagram I and III. To see the equivalence check Example 1.4 and Example 1.12, and note that we define $m_{A\otimes B} = (m_A \otimes m_B) \circ \tau_{23}$ and $\Delta_{C\otimes D} = \tau_{23} \circ (\Delta_C \otimes \Delta_D)$, so

$$(m_B \otimes m_B) \circ \Delta_{B\otimes B} = (m_B \otimes m_B) \circ \tau_{23} \circ (\Delta_B \otimes \Delta_B) = m_{B\otimes B} \circ (\Delta_B \otimes \Delta_B).$$

Similarly, $u_B \colon B \to k$ being a coalgebra map means that

$$(u_B \otimes u_B) \circ \Delta_{B\otimes B} = \Delta_B \circ u_B,$$
$$\varepsilon_k = \varepsilon_B \circ u_B,$$

which is equivalent to diagram II and IV. □

`prop:bialgdual` **Proposition 2.3.** *Let $(B, m, u, \Delta, \varepsilon)$ be a bialgebra with $\dim_k B < \infty$. Then $(B^*, \Delta^*, \varepsilon^*, m^*, u^*)$ is a bialgebra, where we identify $(B \otimes B)^* = B^* \otimes B^*$ (which makes sense for B finite dimensional).*

Proof. Note that when you dualize diagrams I–IV, I and IV are self-dual, and II and III dualize to each other.

For example, dualizing digram II, we get

$$\begin{tikzcd}
k = k^* & B^* \arrow[l, "u_B^*=\varepsilon_{B^*}"'] \\
& (B \otimes B)^* = B^* \otimes B^*, \arrow[u, "\Delta_B^*=m_{B^*}"'] \arrow[ul, "u_{B\otimes B}^*=(u_B\otimes u_B)^*=u_B^*\otimes u_B^*=\varepsilon_{B^*}\otimes\varepsilon_{B^*}"]
\end{tikzcd}$$

which is exatcly diagram III under our identification. □

ex:finmonbialg **Example 2.4.** Let M be a finite monoid and take (kM, m, u) to be the algebra with

$$pq = \begin{cases} p & \text{if } p = q, \\ 0 & \text{if } p \neq q \end{cases} \qquad (p, q \in M),$$
$$1_{kM} = \sum_{p\in M} p.$$

This algebra is clear isomorphic to the algebra $k \times \cdots \times k$ with $|M|$ factors. Consider furthermore the coalgebra $(kM, \Delta, \varepsilon)$ with

$$\Delta(p) = \sum_{qr=p} q \otimes r,$$
$$\varepsilon(p) = \begin{cases} 1 & \text{if } p = 1_M, \\ 0 & \text{if } p \neq 1_M, \end{cases}$$

for $p \in M$. One can now check that this $(kM, m, u, \Delta, \varepsilon)$ is a bialgebra by definition, or we can show that this bialgebra is the dual of the bialgebra from Example 2.2 (when M is finite). ○

Definition 2.5 (Commutative algebra). An algebra A is *commutative* if $ab = ba$ for all $a, b \in A$, or equivalently if $m \circ \tau = m$ as maps $A \otimes A \to A$, where

$$\begin{aligned} \tau\colon A \otimes A &\to A \otimes A \\ a \otimes b &\mapsto b \otimes a. \end{aligned}$$

Definition 2.6 (Cocommutative coalgebra). A coalgebra C *cocommutative* if $\Delta = \tau \circ \Delta$ as maps $A \to A \otimes A$, or equivalently

$$\sum c_{(1)} \otimes c_{(2)} = \Delta(c) = \sum c_{(2)} \otimes c_{(1)}.$$

We say that a bialgebra is commutative, if it is commutative as an algebra, and cocommutative if it is cocommutative as a coalgebra.

Example 2.7. The bialgebra from Example 2.2 is cocommutative, but not commutative in general, and the bialgebra from Example 2.4 is commutative, but not cocommutative in general. ○

2.2 Review of free algebras and presentations

The free (associative) algebra is

$$k\langle x_1, \dots, x_n\rangle = k\text{-span of the words in the } x_i,$$

e.g. for $n = 2$ the words are

$$\{1, x_1, x_2, x_1^2, x_1x_2, x_2x_1, x_2^2, \dots\},$$

with product given by concatenation (extended linearly), e.g.

$$(x_1^2)(x_2x_1) = x_1^2x_2x_1.$$

Universal Property (Free algebra): *Given an algebra A and $a_1, \dots, a_n \in A$ there exists a unique algebra morphism*

$$\begin{aligned} \phi\colon k\langle x_1, \dots, x_n\rangle &\to A \\ x_i &\mapsto a_i. \end{aligned}$$

We can write $k\langle x_1, \dots, x_n\rangle/(r_1, \dots, r_n)$ for the algebra with relations $r_1, \dots, r_n$, i.e. $k\langle x_1, \dots, x_n\rangle/I$ where I is the smallest ideal containing $r_1, \dots, r_n$.

Note that

$$I = \Big\{ \sum_j f_j r_{i_j} g_j \;\Big|\; f_j, g_j \in k\langle x_1, \dots, x_n\rangle \Big\}.$$

ex:quantumbialg **Example 2.8 (Quantum plane).** Consider the algebra

$$A = \frac{k\langle x, y\rangle}{(yx - qxy)}$$

for $0 \neq q \in k$. One can check that A has k-basis $\{x^iy^j \mid i, j \geq 0\}$. We claim that A is a bialgebra with

$$\begin{aligned} \Delta(x) &= x \otimes x, & \varepsilon(x) &= 1, \\ \Delta(y) &= y \otimes 1 + x \otimes y, & \varepsilon(y) &= 0, \end{aligned}$$

i.e. x is a grouplike element, and y is $(1, x)$-primitive.

Proof (that the quantum plane is a bialgebra). First note that, by the universal property of free algebras, there are unique k-algebra homomorphisms

$$\begin{aligned} \widetilde{\Delta}\colon k\langle x, y\rangle &\to k\langle x, y\rangle \otimes k\langle x, y\rangle \\ x &\mapsto x \otimes x \\ y &\mapsto y \otimes 1 + x \otimes y, \\ \varepsilon\colon k\langle x, y\rangle &\to k \\ x &\mapsto 1 \\ y &\mapsto 0. \end{aligned}$$

It remains to check that $\widetilde{\Delta}$ and $\widetilde{\varepsilon}$ induce the maps

$$\begin{aligned} \Delta\colon \frac{k\langle x,y\rangle}{I} &\to \frac{k\langle x,y\rangle}{I} \otimes \frac{k\langle x,y\rangle}{I}, \\ \varepsilon\colon \frac{k\langle x,y\rangle}{I} &\to k, \end{aligned}$$

where $I = (yx - qxy)$. To check this we need to show that

$$\begin{aligned} \widetilde{\Delta} &\subseteq I \otimes k\langle x,y\rangle + k\langle x,y\rangle \otimes I, \\ \widetilde{\varepsilon}(I) &= 0. \end{aligned}$$

We note that

$$\begin{aligned} \widetilde{\Delta}(yx - qxy) &= (y \otimes 1 - x \otimes y)(x \otimes x) - q(x \otimes x)(y \otimes 1 - x \otimes y) \\ &= yx \otimes x - x^2 \otimes yx - qxy \otimes x + qx^2 \otimes xy \\ &= (yx - qxy) \otimes x + x^2 \otimes (-yx + qxy) \in I \otimes k\langle x,y\rangle + k\langle x,y\rangle \otimes I, \end{aligned}$$

and similarly

$$\widetilde{\varepsilon}(yx - qxy) = 0,$$

as we wanted.

Now we have algebra homomorphisms

$$\begin{aligned} \Delta\colon A &\to A \otimes A, \\ \varepsilon\colon A &\to k, \end{aligned}$$

and we want to show that (A, Δ, ε) is a coalgebra. We need to show that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

Here both equations are algebra homomorphisms $A \to A \otimes A \otimes A$ by the above, so they are equal if they agree on generators. For x, the above equation just says

$$x \otimes x \otimes x = x \otimes x \otimes x,$$

which is obviously true, and for y the equation says

$$\begin{aligned} (\Delta \otimes \mathrm{id})(y \otimes 1 + x \otimes y) &= (y \otimes 1 + x \otimes y) \otimes 1 + x \otimes x \otimes y \\ &= y \otimes 1 \otimes 1 + x \otimes (y \otimes 1 + x \otimes y) \\ &= (\mathrm{id} \otimes \Delta)(x \otimes 1 + x \otimes y,) \end{aligned}$$

so coassociativity holds. Similarly,

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta, \mathrm{id}, (\mathrm{id} \otimes \varepsilon) \circ \Delta$$

are all algebra homomorphisms, so again we just have to check that these are all equal on x, y. We see that

$$\begin{aligned}(\varepsilon \otimes \mathrm{id})(\Delta(y)) &= \varepsilon(y)1 + \varepsilon(x)y = 0 + y = y = \mathrm{id}(y)\\ &= y + 0 = y\varepsilon(1) + x\varepsilon(y) = (\mathrm{id} \otimes \varepsilon)(\Delta(y)),\end{aligned}$$

as we wanted. Hence $(A, m, u, \Delta, \varepsilon)$ is a bialgebra. □

We call this bialgebra the *quantum plane*. ○

Example 2.9 (Sweedler/Taft algebra). Consider

$$B = \frac{k\langle x, y\rangle}{(yx + xy, x^2 - 1, y)}$$

with

$$\begin{aligned}\Delta(x) &= x \otimes x, & \varepsilon(x) &= 1,\\ \Delta(y) &= y \otimes 1 + x \otimes y & \varepsilon(y) &= 0.\end{aligned}$$

We claim that this is a finite dimensional (with $\dim_k B = 4$) bialgebra which is neither commutative nor cocommutative.

We first note that B has basis $\{1, x, y, xy\}$ (easy to see this is a spanning set, and linear independence is left to the reader). Now, since

$$A = \frac{k\langle x, y\rangle}{(yx + xy)}$$

is a bialgebra, it is enough to check that the ideal of A generated by $(x^2-1, y^2) =:$ J is a *biideal* of A (i.e. an ideal and coideal).

To see that J is a coideal, we note that

$$\begin{aligned}\Delta(x^2 - 1) &= x^2 \otimes x^2 - 1 \otimes 1\\ &= (x^2 - 1) \otimes x^2 + 1 \otimes (x^2 - 1) \in J \otimes A + A \otimes J,\end{aligned}$$

and similarly

$$\begin{aligned}\Delta(y^2) &= (y \otimes 1 + x \otimes y)(y \otimes 1 + x \otimes y)\\ &= y^2 \otimes 1 + yx \otimes y + xy \otimes y + x^2 \otimes y^2\\ &= \underbrace{y^2 \otimes 1 + x^2 \otimes y^2}_{\in J\otimes A + A\otimes J} + \underbrace{(yx + xy)}_{=0} \otimes y.\end{aligned}$$

Also

$$\begin{aligned}\varepsilon(x^2 - 1) &= 1^2 - 1 = 0,\\ \varepsilon(y^2) &= 0^2 = 0,\end{aligned}$$

so J is a biideal. Hence $B = A/J$ is a bialgebra. ○

Chapter 3

Hopf Algebras

cha:hopfalgs

Definition 3.1 (Convolution algebra). Let C be a coalgebra and A an algebra. The *convolution algebra* is

$$\mathrm{Hom}_k(C, A)$$

with product $f * g$ given by

$$[f * g](c) = m \circ (f \otimes g) \circ \Delta(c) = \sum f(c_{(1)})g(c_{(2)}).$$

This is an algebra with with identity element $u \circ \varepsilon$.

To check that $u \circ \varepsilon$ is the identity, note that

$$\begin{aligned}
[(u \circ \varepsilon) * g](c) &= \sum u \circ \varepsilon(c_{(1)})g(c_{(2)}) \\
&= \sum \varepsilon(c_{(1)})g(c_{(2)}) && \text{(identifying } k \text{ with } u(k) \subseteq A) \\
&= g\Big(\sum \varepsilon(c_{(1)})c_{(2)}\Big) && \text{(since } g \text{ is linear)} \\
&= g(c) && \text{(counit axiom),}
\end{aligned}$$

so $(u \circ \varepsilon) * g = g$. Similarly $g * (u \circ \varepsilon) = g$, and $*$ is associative since

$$(f * g) * h = \sum f(c_{(1)})g(c_{(2)})h(c_{(3)}) = f * (g * h)$$

using coassociativity.

Definition 3.2 (Hopf algebra). A bialgebra H is a *Hopf algebra* if, in the convolution algebra $\mathrm{Hom}_k(H, H)$, $\mathrm{id}_H \colon H \to H$ has an inverse $S \colon H \to H$ under convolution:

$$S * \mathrm{id}_H = u \circ \varepsilon = \mathrm{id}_H * S.$$

In this case, S is called the *antipode* of H.

Remarks.

(1) A Hopf algebra is not an additional structure on a bialgebra.

(2) S, if it exists, is unique (being an inverse element in an algebra).

(3) In Sweedler notation S satisfies for all $h \in H$

$$\sum S(h_{(1)})h_{(2)} = \varepsilon(h) = \sum h_{(1)}S(h_{(2)}). \qquad \triangle$$

Example 3.3. Let G be a group. Then the bialgebra kG, where (kG, m, u) is a group algebra and $(kG, \Delta, \varepsilon)$ is a grouplike coalgebra, is a Hopf algebra. To see this, consider

$$\begin{aligned} S\colon kG &\to kG \\ G \ni g &\mapsto g^{-1} \end{aligned}$$

and note that for $g \in G$

$$\begin{aligned} 1 = \varepsilon(g) &= S(g)g = g^{-1}g = 1 \\ &= gS(g) = gg^{-1} = 1. \end{aligned} \qquad \bigcirc$$

Proposition 3.4. *Let H be a Hopf algebra with antipode S. Then:*

(1) S is an algebra anti-homomorphism, i.e.

$$S(ab) = S(b)S(a),$$

or

$$S \circ m = m \circ (S \otimes S) \circ \tau \colon H \otimes H \to H.$$

(2) S is a colagebra anti-homomorphism, i.e.

$$\Delta(S(a)) = \sum S(a_{(2)}) \otimes S(a_{(1)}),$$

or

$$\Delta \circ S = \tau \circ (S \otimes S) \circ \Delta \colon H \to H \otimes H.$$

Proof.

(1) Consider the convolution algebra

$$R = \mathrm{Hom}_k(H \otimes H, H).$$

We note that

$$
\begin{aligned}
&[(S \circ m) * m](a \otimes b) \\
&= \sum (S \circ m)(a \otimes b)_{(1)} m((a \otimes b)_{(2)}) \\
&= \sum (S \circ m)(a_{(1)} \otimes b_{(2)}) m(a_{(2)} \otimes b_{(2)}) && \text{(by definition of } \Delta_{H \otimes H}) \\
&= \sum S(a_{(1)} b_{(2)}) a_{(2)} b_{(2)} \\
&= \sum S((ab)_{(1)})(ab)_{(2)} && (\Delta_H \text{ is an algebra homomorphism}) \\
&= \varepsilon(ab) \\
&= \varepsilon(a)\varepsilon(b) \\
&= \varepsilon_{H \otimes H}(a \otimes b) && \text{(by definition of } \varepsilon_{H \otimes H}),
\end{aligned}
$$

so $(S \circ m) * m = \varepsilon$ in R. Similarly

$$
\begin{aligned}
&m * (m \circ (S \otimes S) \circ \tau)(a \otimes b) \\
&= \sum m((a \otimes b)_{(1)}) m \circ (S \otimes S) \circ \tau((a \otimes b)_{(2)}) \\
&= \sum m(a_{(1)} \otimes b_{(1)}) m \circ (S \otimes S) \circ \tau(a_{(2)} \otimes b_{(2)}) \\
&= \sum a_{(1)} b_{(1)} S(b_{(1)}) S(a_{(1)}) \\
&= \sum \varepsilon(b) a_{(1)} S(a_{(2)}) && \text{(by axiom of } S) \\
&= \varepsilon(a)\varepsilon(b) && \text{(by axiom of } S) \\
&= \varepsilon_{H \otimes H}(a \otimes b),
\end{aligned}
$$

so $m * (m \circ (S \otimes S) \circ \tau) = \varepsilon$ in R. Thus

$$m \circ (S \otimes S) \circ \tau = S \circ m,$$

since both are inverse to m under convolution.

(2) Same idea for the argument, but this time using

$$\mathrm{Hom}_k(H, H \otimes H). \qquad \square$$

Lemma 3.5. *Let H be a Hopf algebra and define*

$$G \coloneqq \{g \in H \setminus \{0\} \mid \Delta(g) = g \otimes g\},$$

the set of grouplike elements in H. Then every $g \in G$ is a unit and G is a group under multiplication.

Proof. It's easy to see that $\varepsilon(g) = 1$ for all $g \in G$ (cf. Homework 1).[1] Now

$$\varepsilon(g) = 1 = S(g)g = gS(g)$$

for $g \in G$, so g is a unit and $S(g) = g^{-1}$. Also, if $g, h \in G$, then $gh \in G$ since Δ is an algebra map. Finally, since S is a coalgebra anti-homomorphism

$$\Delta(g^{-1}) = \Delta(S(g)) = \sum S(g) \otimes S(g) = g^{-1} \otimes g^{-1},$$

so $g^{-1} \in G$ for $g \in G$. Hence G is a group. □

Corollary 3.6. *If M is a monoid, then the monoid bialgebra kM (with grouplike coalgebra structure) is a Hopf algebra if and only if M is a group.*

Proof. The elements in M are grouplike, so if kM is a Hopf algebra, then M consists of units. In fact $M = G$. The converse statement we have already shown. □

Example 3.7 (Quantum plane). The quantum plane

$$k_q[x, y] = \frac{k\langle x, y\rangle}{(yx - qxy)} \qquad (0 \neq q \in k),$$

with

$$\begin{aligned} \Delta(x) &= x \otimes x, & \varepsilon(x) &= 1, \\ \Delta(y) &= y \otimes 1 + x \otimes y, & \varepsilon(y) &= 0, \end{aligned}$$

is a bialgebra, but not a Hopf algebra. This is by the same reason as above; x is grouplike but not a unit. ○

Example 3.8 (Taft algebra). The Taft algebra

$$\frac{k\langle x, y\rangle}{(yx + xy, x^2 - 1, y^2)}$$

is a Hopf algebra with

$$\begin{aligned} S(x) &= x^{-1} = x, \\ S(y) &= -xy = yx. \end{aligned}$$ ○

Proposition 3.9. *If H is a finite dimensional Hopf algebra $H = (H, m, u, \Delta, \varepsilon, S)$, then so is H^*, where*

$$S_{H^*} = S^*.$$

[1] Also, the elements of G are linearly independent over k, cf. Homework 1.

Proof. We saw in Proposition 2.3 that $(H^*, \Delta^*, \varepsilon^*, m^*, u^*)$ is a bialgebra (since $\dim_k H < \infty$). So we just need to show that S^* is an antipode. Since

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta,$$

dualizing, we get that

$$\Delta^* \circ (S^* \otimes \mathrm{id}) \circ m^* = \varepsilon^* \circ u^* = \Delta^* \circ (\mathrm{id} \otimes S^*) \circ m^*.$$

Hence S^* is indeed an antipode. □

Definition 3.10 (Hopf algebra morphism). A linear map $f \colon H \to H'$ between Hopf algebras is a morphism of Hopf algebras if it is a bialgebra morphism, and

$$f \circ S_H = S_{H'} \circ f.$$

Example 3.11. If H is the 4 dimensional Taft algebra, you can check that $H^* \cong H$ as Hopf algebras. (Exercise.) ○

3.1 Universal enveloping algebras of Lie algebras

Definition 3.12 (Lie algebra). A *Lie algebra* is a vector space L over k with a bilinear product

$$\begin{aligned} L \times L &\to L \\ (x, y) &\mapsto [x, y] \end{aligned}$$

such that

$$[x, x] = 0[x, y] \quad = -[y, x][x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{(Jacobi identity)}$$

for all $x, y, z \in L$.

Now assume that L is a finite dimensional (over k) Lie algebra. If L has basis $\{x_1, \dots, x_n\}$, then we define the *universal enveloping algebra of* L to be

$$U(L) = \frac{k\langle x_1, \dots, x_n\rangle}{(x_j x_i - x_i x_j - [x_i, x_j] \mid 1 \leq i < j \leq n)}.$$

Example 3.13. $L = kx + ky$ with $[x, y] = x = -[y, x]$ is a Lie algebra, and

$$U(L) \cong \frac{k\langle x, y\rangle}{(yx - xy - x)}.$$ ○

Example 3.14 (Abelian Lie algebra). $L = kx_1 + \cdots + kx_n$ with $[x_i, x_j] = 0$ for all i, j is an Abelian Lie algebra, and

$$U(L) = \frac{k\langle x_1, \ldots, x_n\rangle}{(x_j x_i - x_i x_j)} \cong k[x_1, \ldots, x_n]. \qquad \bigcirc$$

thm:PBW **Theorem 3.15 (Poincaré-Birkhoff-Witt [PBW]).** *If L is a Lie algebra with basis $\{x_1, \ldots, x_n\}$, then $U(L)$ has k-basis*

$$\{x_1^{i_1} \cdots x_n^{i_n} \mid i_j \geq 0\}.$$

Remark. A module over L is the same as a module over $U(L)$. $\triangle$

Example 3.16 (Universal enveloping algebra). Let L be a Lie algebra with basis $x_1, \ldots, x_n$.[2] Then $U(L)$ is a Hopf algebra with

$$\begin{aligned} \Delta(x_i) &= x_i \otimes 1 + 1 \otimes x_i && \text{(i.e. } x_i \text{ is } (1,1)\text{-primitive)}, \\ \varepsilon(x_i) &= 0, \\ S(x_i) &= -x_i. \end{aligned}$$

Actually, $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in L$, and similarly $\varepsilon(x) = 0$ and $S(x) = -x$ for all $x \in L$.

Showing that $U(L)$ is a bialgebra is similar to the quantum plane example (cf. Example 2.8). Similarly, to check that S is an antipode, we only need to check on the generating set of the algebra, so

$$S(x_i)1 + S(1)x_i = -x_i + x_i = 0 = \varepsilon(x_i),$$

and similarly

$$x_i S(1) + 1S(x_i) = 0. \qquad \bigcirc$$

Example 3.17. The Lie algebra $L = kx$ has $[x, x] = 0$, $U(L) = k[x]$, and

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x, \\ \varepsilon(x) &= 0, \\ S(x) &= -x. \end{aligned}$$

It is interesting to compute

$$\Delta(x^n) = (\Delta(x))^n = (x \otimes 1 + 1 \otimes x)^n = \sum_{i=0}^{n} \binom{n}{i} (x \otimes 1)^i (1 \otimes x)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}.$$

$\bigcirc$

[2] We don't need $\dim_k L < \infty$ for this, but it simplifies the notation.

3.2 Coordiante rings of algebraic groups

Let R be a commutative finitely generated k-algebra, say

$$R = \frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)},$$

where $k = \overline{k}$.

Let

$$X = \max \operatorname{Spec} R = \{\text{maximal ideals of } R\},$$

which is a *closed* subset of affine n-space $\mathbb{A}^n = k^n$, where

$$\begin{aligned} \max \operatorname{Spec} k[x_1, \ldots, x_n] &= \mathbb{A}^n = k^n \\ (x_1 - a_1, \ldots, x_n - a_n) &\mapsto (a_1, \ldots, a_n) \end{aligned}$$

for $a_i \in k$ by the Nullstellensatz, and

$$\max \operatorname{Spec} \frac{k[x_1, \ldots, x_n]}{(f_1, \ldots, f_m)} \to X = \{(a_1, \ldots, a_n) | f_j(a_1, \ldots, a_n) = 0 \text{ for all } j\} \subseteq \mathbb{A}^n.$$

X as above is an affine closed set.

An affine algebraic group is such an X which is also a group where

$$\begin{aligned} p \colon X \times X &\to X, & v \colon X &\to X, & 1_X \colon \operatorname{Spec} k &\to X, \\ (x, y) &\mapsto xy & x &\mapsto x^{-1} \end{aligned}$$

where p and v are regular maps of X (given by ratios of polynomials in words).

Claim: $R =$ coordinate ring of X is a Hopf algebra.

From algebraic geometry, affine varieties and regular maps are dual to commutative k-algebras and algebra maps. Also product of varieties correspond to tensor product of algebras. Dualizing the multiplication map p and inverse map v gives algebra maps

$$\begin{aligned} \Delta \colon R &\to R \otimes_k R & &(\text{dual of } p), \\ S \colon R &\to R & &(\text{dual of } v), \\ \varepsilon \colon R &\to k & &(\text{dual of } 1_X). \end{aligned}$$

(Proof omitted — concentrate on examples.)

Example 3.18. $\mathbb{A}^1 = k = \max \operatorname{Spec} k[x]$ is an algebraic group under

$$\begin{aligned} p \colon \mathbb{A}^1 \times \mathbb{A}^1 &\to \mathbb{A}^1, & v \colon \mathbb{A}^1 &\to \mathbb{A}^1 & 1_{\mathbb{A}^1} &= 0. \\ (a, b) &\mapsto a + b & a &\mapsto -a \end{aligned}$$

The corresponding Hopf structure on $k[x]$ is

$$\begin{aligned} \Delta \colon k[x] &\to k[x] \otimes k[x], & S \colon k[x] &\to k[x], & \varepsilon \colon k[x] &\to k. \\ x &\mapsto x \otimes 1 + 1 \otimes x & x &\mapsto -x & x &\mapsto 0 \end{aligned}$$

This is the same as $U(L)$ for a 1 dimensional Lie algebra L. ○

Example 3.19.

$$X = \mathbb{A}^1 - \{0\} = k - \{0\} = \max\operatorname{Spec} k[x, x^{-1}] \qquad \left(k[x, x^{-1}] \cong \frac{k[x,y]}{(yx-1)}\right)$$

is an algebraic group with

$$\begin{aligned} p\colon X \times X &\to X, & v\colon X &\to X, & 1_X &= 1. \\ (a,b) &\mapsto ab & a &\mapsto a^{-1} \end{aligned}$$

The corresponding Hopf structure on $R = k[x, x^{-1}]$ is

$$\begin{aligned} \Delta\colon R &\to R \otimes R, & S\colon R &\to R, & \varepsilon\colon R &\to k. \\ x &\mapsto x \otimes x & x &\mapsto x^{-1} & x &\mapsto 1 \end{aligned}$$

In fact this is $k\mathbb{Z}$ up to isomorphism, since

$$G = \text{grouplike elements of } R = \{x^i \mid i \in \mathbb{Z}\} \cong (\mathbb{Z}, +).$$

○

Example 3.20. Consider

$$X = \mathrm{SL}_2(k) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; ad - bc = 1 \right\} \subseteq k^4 = \mathbb{A}^4,$$

i.e.

$$X = \max\operatorname{Spec} R$$

where

$$R = \frac{k[x_{11}, x_{12}, x_{21}, x_{22}]}{(x_{11}x_{22} - x_{12}x_{21} - 1)},$$

is an algebraic group with

$$\begin{aligned} p\colon X \times X &\to X, & v\colon X &\to X, & 1_X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \\ (A,B) &\mapsto AB & A &\mapsto A^{-1} \end{aligned}$$

In coordinates we have e.g.

$$v\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and similarly p can be described by polynomials in the entries.

The Hopf structure on R is

$$\begin{aligned} \Delta\colon R &\to R \otimes R, & S\colon R &\to R, & \varepsilon\colon R &\to k. \\ x_{ij} &\mapsto \sum_{\ell=1}^{n} x_{i\ell} \otimes x_{\ell j} & x_{11} &\mapsto x_{22} & x_{ij} &\mapsto \delta_{ij} \\ & & x_{22} &\mapsto x_{11} & & \\ & & x_{12} &\mapsto -x_{12} & & \\ & & x_{21} &\mapsto -x_{21} & & \end{aligned}$$

○

3.3 Modules and comodules

Let A be a k-algebra. Recall that a left A-module is an Abelian group M with a bilinear map

$$\begin{aligned} \mu\colon A \times M &\to M \\ (a, x) &\mapsto a \,.\, x = ax \end{aligned}$$

such that

$$\begin{aligned} 1x &= x, \\ a(bx) &= (ab)x \end{aligned}$$

for all $x \in M$ and $a, b \in A$. Note that M is also a k-vector space since $k \subseteq A$. We write M as (M, μ).

Note furthermore, if (A, m, u) is the algebra, then (M, μ) — where we abuse notation and write μ for the map $\mu\colon A \otimes M \to M$ — is a module over A if and only if the following commute.

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "m_A \otimes \mathrm{id}_M"] \arrow[d, "\mathrm{id}_A \otimes M"'] & A \otimes M \arrow[d, "\mu"] \\ A \otimes M \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} k \otimes M \arrow[rr, "u \otimes \mathrm{id}_M"] \arrow[dr, equal] & & A \otimes M \arrow[dl, "\mu_M"] \\ & M & \end{tikzcd}$$

Dualizing we get:

Definition 3.21 (Right comodule). Let C be a coalgebra over k, $C = (C, \Delta, \varepsilon)$. A *right comodule over* C is a vector space N and a linear map $\rho\colon N \to N \otimes C$ such that the following diagrams commute:

$$\begin{tikzcd} N \arrow[r, "\rho"] \arrow[d, "\rho"'] & N \otimes C \arrow[d, "\mathrm{id}_N \otimes \Delta"] \\ N \otimes C \arrow[r, "\rho \otimes \mathrm{id}_C"'] & N \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} N \arrow[rr, "\rho"] \arrow[dr, equal] & & N \otimes C \arrow[dl, "\mathrm{id}_N \otimes \varepsilon"] \\ & N \otimes k & \end{tikzcd}$$

Example 3.22. Let (C, Δ, ε) be a coalgebra. C is a right comodule over itself with $\rho = \Delta\colon C \to C \otimes C$, since the needed diagrams are part of axioms of the coalgebra. Similarly, C is a left comodule over itself using Δ. ○

Example 3.23. Let (C, Δ, ε) be a coalgebra. Suppose $I \subseteq C$ is a subspace I such that

$$\Delta(I) \subseteq I \otimes C,$$

i.e. I is a *right coideal.* Then $(I, \Delta|_I)$ is a comodule and a subcomodule of C. ○

Example 3.24. Consider $C = k[x]$ with

$$\Delta(x^n) = \sum_{i=0}^{n} x^i \otimes x^{n-i},$$
$$\varepsilon(x^n) = \delta_{0n}.$$

(C, Δ, ε) is a coalgebra.

Let N be a k-vector space with a linear transformation $\phi\colon N \to N$ such that ϕ is locally nilpotent, i.e. for all $n \in N$, $\phi^i(n) = 0$ for all $i \gg 0$. Now define

$$\rho\colon N \to N \otimes k[x]$$
$$n \mapsto \sum_{s \geq 0} \phi^s(n) \otimes x^s,$$

where $\phi^0 = \mathrm{id}_N$. Local nilpotence implies that ρ is well-defined. To check that this a comodule, we note that

$$(\mathrm{id} \otimes \Delta) \circ \rho(n) = (\mathrm{id} \otimes \Delta)\Big(\sum_s \phi^s(n) \otimes x^s\Big) = \sum_s \sum_{i=0}^{s} \phi^s(n) \otimes x^i \otimes x^{s-i}$$

and

$$(\rho \otimes \mathrm{id})\Big(\sum_s \phi^s(n) \otimes x^s\Big) = \sum_s \sum_t \phi^t(\phi^s(n)) \otimes x^t \otimes x^s = \sum_r \sum_t \phi^{r+t}(n) \otimes x^t \otimes x^r,$$

where these are equal with $t = i$ and $r = s - i$. Furthermore

$$(\mathrm{id} \otimes \varepsilon)(\rho(n)) = \sum_s \phi^s(n) \otimes \varepsilon(x^s) = \phi^0(n) = n,$$

so (N, ρ) is indeed a comodule. ○

Remark. All right comodules over $C = k[x]$ as above have this form. △

Definition 3.25. Let N and P be two right comodules over C. Then $\psi\colon N \to P$ is a comodule morphism if

$$\begin{array}{ccc} N & \xrightarrow{\psi} & P \\ {\scriptstyle \rho_N}\downarrow & & \downarrow{\scriptstyle \rho_P} \\ N \otimes C & \xrightarrow[\psi \otimes \mathrm{id}_C]{} & P \otimes C \end{array}$$

commutes.

All standard results (e.g. about kernels) still works in this setting.

3.4 Duality between right C-comodules and certain left C^*-modules

Definition 3.26 (Closed and cofinite subspaces). Let V be a vector space over k. Let $V^* = \mathrm{Hom}_k(V, k)$. A subspace $X \subseteq V^*$ is called *closed* if $X = W^\perp$ for some subspace $W \subseteq V$, i.e.

$$X = \{f \in V^* \mid f(w) = 0 \text{ for all } w \in W\}.$$

$X \subseteq V^*$ is *cofinite* if $\dim_k V^*/X < \infty$.

$X \subseteq V^*$ is cofinite and closed if and only if $X = W^\perp$ for some finite dimensional subspace $W \subseteq V$.

Definition 3.27 (Rational C^*-modules). Let C be a coalgebra, and so C^* is an algebra. A left C^*-module M is *rational* if for all $m \in M$,

$$am_{C^*}(m)$$

is cofinite and closed subspace of C^*. (Note that it is always a left ideal of C^*.)

Remark. If M is rational, then

$$C^* \,.\, m \cong C^*/am_{C^*}(m)$$

as left modules. So $C^* \,.\, m$ is finite dimensional. So M is the union of finite dimensional submodules, i.e. M is locally finite. △

Notation of Sweedler for comodules

If (M, ρ) is a right C-module, we write

$$\rho(m) = \sum \underbrace{m_{(0)}}_{\in M} \otimes \underbrace{m_{(1)}}_{\in C}.$$

For a left C-comodule (M, ρ), we write

$$\rho(m) = \sum \underbrace{m_{(-1)}}_{\in C} \otimes \underbrace{m_{(0)}}_{\in M}.$$

Duality results

Theorem 3.28. *Let C be a coalgebra and let C^* be the dual algebra.*

(1) If (N, ρ) is a right C-comodule, then (N, μ) is a left C^*-module which is rational, where

$$\begin{aligned} \mu\colon C^* \otimes N \xrightarrow{\mathrm{id}_C \otimes \rho} C^* \otimes N \otimes C &\longrightarrow N \\ f \otimes n \otimes c &\longmapsto f(c)n. \end{aligned}$$

(2) *If (N, μ) is a rational left C^*-module, then there is a natural right C-comodule structure (N, ρ).*

(3) *These processes are inverse.*

Corollary 3.29. *Right C-comodules are in bijection with rational left C^*-modules.*

If $\dim_k C < \infty$, then all C^*-modules are rational.

Proof (Sketch of theorem part (1)). We want to show that (N, μ) is a left C^*-module. For $f, g \in C^*$ and $n \in N$ we note that the map $\mu \circ (\mathrm{id} \otimes \mu)$ takes

$$\begin{aligned} f \otimes g \otimes n &\mapsto \sum f \otimes g \otimes n_{(0)} \otimes n_{(1)} \\ &\mapsto \sum g(n_{(1)}) f \otimes n_{(0)} \\ &\mapsto \sum g(n_{(1)}) f \otimes n_{(0)(0)} \otimes n_{(0)(1)} = g(n_{(2)}) \otimes n_{(0)} \otimes n_{(1)} \\ &\mapsto \sum g(n_{(2)}) f(n_{(1)}) n_{(0)}, \end{aligned}$$

while the map $\mu \circ (m_{C^*} \otimes \mathrm{id})$ takes

$$\begin{aligned} f \otimes g \otimes n &\mapsto fg \otimes n \\ &\mapsto \sum fg \otimes n_{(0)} \otimes n_{(1)} \\ &\mapsto \sum (fg)(n_{(1)}) \otimes n_{(0)} = \sum f(n_{(1)(1)}) g(n_{(1)(2)}) n_{(0)} \\ &\qquad\qquad\qquad\qquad\qquad\; = \sum f(n_{(1)}) g(n_{(2)}) n_{(0)}. \end{aligned}$$

Thus we see that

$$\mu \circ (\mathrm{id} \otimes \mu) = \mu \circ (m_{C^*} \otimes \mathrm{id}).$$

Also $1_{C^*} = \varepsilon$, so μ takes

$$\varepsilon \otimes n \mapsto \sum \varepsilon \otimes n_{(0)} \otimes n_{(1)} \mapsto \sum \varepsilon(n_{(1)}) n_{(0)} = \sum n_{(0)} \varepsilon(n_{(1)}) = n$$

by comodule axioms.

Finally to see that N is rational, we note that if $n \in N$, then

$$\rho(n) = \sum_{i=1}^{q} n_i \otimes c_i$$

for some $n_i \in N$ and $c_i \in C$. So, if $W = kc_1 + \cdots + kc_q$, then $I := W^\perp$ is closed and cofinite, and $\mu(I \otimes n) = 0$. $\square$

Remark. We won't prove part (2), but note that if C is finite dimensional, then (2) is proved similarly. Given (N,μ) a left C^*-module, we define (N,ρ) a right C-comodule by

$$\begin{aligned} \rho\colon N &\longrightarrow C^*\otimes N\otimes C \xrightarrow{\mu\otimes \mathrm{id}_C} N\otimes C, \\ n &\longmapsto \sum c_i^*\otimes n\otimes c_i \end{aligned}$$

where $\{c_i\}$ is any basis of C. △

Example 3.30. Consider the coalgebra $C = k[x]$ with

$$\begin{aligned} \Delta(x^n) &= \sum_{i=0}^{n} x^i\otimes x^{n-i}, \\ \varepsilon(x^n) &= \delta_{0n}. \end{aligned}$$

Let C^* be the dual algebra, and recall that $C^*\cong k[\![z]\!]$ (cf. Homework 1) via

$$\begin{aligned} C^* = \mathrm{Hom}_k(C,k) &\to k[\![z]\!] \\ f &\mapsto \sum_{i\geq 0} f(x^i)z^i. \end{aligned}$$

Consider a rational left C^*-module N, and let

$$\begin{aligned} \phi\colon N &\to N \\ n &\mapsto z\,.\,n. \end{aligned}$$

Since N is rational, it is locally finite. This implies for $n\in N$, $k[\![z]\!]n$ is finite dimensional, so $z^s\,.\,n = 0$ for $s\gg 0$. Hence ϕ is locally nilpotent. Also, (z^s) is closed (and cofinite) in $k[\![z]\!]$.

Now check that the corresponding right C-comodule on N is the one we defined

$$\rho(n) = \sum_s \phi^s(n)\otimes x^s.$$ ○

3.5 Monoidal structure on modules

Proposition 3.31. *Let B be a bialgebra, and let M and N be left B-modules. Then $M\otimes_k N$ is again a left B-module where*

$$b\,.\,(m\otimes n) = \sum b_{(1)}\,.\,m\otimes b_{(2)}\,.\,n.$$

Proof. $\Delta\colon B\to B\otimes B$ is a map of algebras. Since M and N are B-modules, $M\otimes_k N$ is a $B\otimes B$-module with

$$(a\otimes b)\,.\,(m,n) = (a\,.\,m\otimes b\,.\,n).$$

Now pullback via Δ to get a B-module structure on $M \otimes N$. Also note that $\Delta(1) = 1 \otimes 1$, so

$$1 \,.\, (m \otimes n) = m \otimes n. \qquad \square$$

You can formulate this as saying that the category of left B-modules is a *monoidal category*.

Example 3.32. Let G be a group, and let M and N be representations of G (i.e. kG-modules). Then $M \otimes N$ is also a representation, where

$$g \,.\, (m \otimes n) = g \,.\, m \otimes g \,.\, n$$

for all $g \in G$. ○

Example 3.33. Let $U(L)$ be the universal enveloping algebra of a Lie algebra L. Let M and N be representations of L (so modules over $U(L)$). Then $M \otimes N$ is again a representation, where

$$x \,.\, (m \otimes n) = (x \,.\, m \otimes n) + (m \otimes x \,.\, n)$$

for $x \in L$. (Recall that $\Delta(x) = x \otimes 1 + 1 \otimes x$.) ○

3.6 Hopf modules

If B is a bialgebra, we defined (left or right) B-modules, and (left or right) B-comodules. It is natural to define a structure that is both a module and a comodule with some added axioms.

Definition 3.34 (Hopf module). We say M is a (right,right) Hopf module over B if

(1) (M, μ) is a right B-module (where $\mu\colon M \otimes B \to M$),

(2) (M, ρ) is a right B-comodule (where $\rho\colon M \to M \otimes B$),

and

(3) ρ is a right B-module map

or

(3') μ is a right B-comodule map.

In this definition (3) and (3') are equivalent.

Recall in (3) that $M \otimes B$ is a right B-module with

$$(m \otimes b) \,.\, c = \sum m \,.\, c_{(1)} \otimes b \,.\, c_{(2)}.$$

Also in (3'), $M \otimes B$ is a right B-comodule with

$$\rho_{M\otimes B}(m \otimes b) = \sum m_{(0)}b_{(1)} \otimes m_{(1)}b_{(2)}.$$

Note, (3) says that

$$\begin{tikzcd} M \otimes B \arrow[r, "\rho\otimes \mathrm{id}_B"] \arrow[d, "\mu_M"'] & M \otimes B \otimes B \arrow[d, "\mu_{M\otimes}\ \mathrm{id}_B"] \\ M \arrow[r, "\rho"'] & M \otimes B \end{tikzcd}$$

commutes. In Sweedler notation

$$\begin{aligned} \rho(mb) &= \sum (mb)_{(0)} \otimes (mb)_{(1)} \\ &= \sum m_{(0)}b_{(1)} \otimes m_{(1)}b_{(2)} \\ &= \sum \mu_{M\otimes B}(m_{(0)} \otimes m_{(1)} \otimes b). \end{aligned}$$

Example 3.35. B itself is a (right,right) Hopf module with

$$\begin{aligned} \mu = m &\colon B \otimes B \to B, \\ \rho = \Delta &\colon B \to B \otimes B. \end{aligned}$$

○

Example 3.36. Similarly, $\bigoplus_{i\in I} B$ is a Hopf module. This is a *free* Hopf module. ○

Definition 3.37. Let M be a Hopf module over a bialgebra B.

$$M^{\mathrm{coinv}} = \{m \in M \mid \rho(m) = m \otimes 1_B\}$$

is called the *coinvariants of* M and is a k-subspace.

Theorem 3.38. *Let H be a Hopf algebra and let M be a Hopf module over H. Then*

$$M \cong M^{\mathrm{coinv}} \otimes_k H$$

as right Hopf modules, where $M^{\mathrm{coinv}} \otimes H$ is a right module and comodule using the second coordinate,

$$\begin{aligned} (m \otimes h) \,.\, g &= m \otimes hg, \\ \rho(m \otimes h) &= \sum m \otimes h_{(1)} \otimes h_{(2)}, \end{aligned}$$

i.e. $M^{\mathrm{coinv}} \otimes H$ is free of rank $\dim_k M^{\mathrm{coinv}}$.

Proof. Define

$$\begin{aligned}\alpha\colon M^{\mathrm{coinv}}\otimes_k H &\to M\\ m\otimes h&\mapsto mh,\\ \beta\colon M&\to M^{\mathrm{coinv}}\otimes_k H\\ m&\mapsto \sum m_0 S(m_1)\otimes m_2.\end{aligned}$$

Remark. Here we write

$$(\rho\otimes\mathrm{id})\circ\rho(m)=\sum m_0\otimes m_1\otimes m_2$$

with no parentheses around indices now. △

Claim: α and β are inverse bijections and maps of Hopf modules.
Step 1: β is well-defined.
For $m\in M$ with $\rho(m)=\sum m_0\otimes m_1$, we have

$$\begin{aligned}\rho\Big(\sum m_0S(m_1)\Big)&=\sum (m_0S(m_1))_0\otimes(m_0S(m_1))_1\\ &=\sum m_0S(m_2)_1\otimes m_1S(m_2)_2 && \text{(axiom (3) of Hopf modules)}\\ &=\sum m_0S(m_3)\otimes m_1S(m_2) && \text{(since } S \text{ is an anti coalgebra map)}\\ &=\sum m_0S(m_2)\otimes\varepsilon(m_1)\\ &=\sum m_0S(\varepsilon(m_1)m_2)\otimes 1 && \text{(by linearity)}\\ &=\sum m_0S(m_1)\otimes 1 && \text{(by the counit axiom).}\end{aligned}$$

This shows that

$$\mu\circ(\mathrm{id}\otimes S)\circ\rho(m)=\sum m_0S(m_1)\otimes 1,$$

and thus

$$\sum m_0S(m_1)\in M^{\mathrm{coinv}}.$$

Writing β as $\sum m_{00}S(m_{01})\otimes m_1$, we see that

$$\beta(m)\in M^{\mathrm{coinv}}\otimes_k H.$$

Step 2: $\alpha\circ\beta=\mathrm{id}_M$.
We see that

$$\begin{aligned}\alpha\circ\beta(m)&=\alpha\Big(\sum m_0S(m_1)\otimes m_2\Big)\\ &=\sum m_0S(m_1)m_2\\ &=\sum m_0\varepsilon(m_1) && \text{(by axiom of } S)\\ &=m && \text{(by comodule axioms).}\end{aligned}$$

Step 3: $\beta \circ \alpha = \mathrm{id}_{M^{\mathrm{coinv}} \otimes_k H}$.

We see that

$$\begin{aligned}(\beta \circ \alpha)(m \otimes h) &= \beta(mh) \\ &= \sum (mh)_0 S\big((mh)_1\big) \otimes (mh)_2 \\ &= \sum m_0 h_1 S(m_1 h_2) \otimes m_2 h_3 \qquad \text{(by Hopf module axiom (3))}.\end{aligned}$$

Since $m \in M^{\mathrm{coinv}}$, $\sum m_0 \otimes m_1 \otimes m_2 = m \otimes 1 \otimes 1$, and thus continuing the calculation

$$\begin{aligned}&= \sum m h_1 S(h_2) \otimes h_3 \\ &= \sum m \varepsilon(h_1) \otimes h_2 \\ &= \sum m \otimes \varepsilon(h_1) h_2 \\ &= m \otimes h \qquad \text{(by comodule axioms)}.\end{aligned}$$

Step 4: α is a Hopf module map.

To see that α is a module map, we note that

$$\alpha((m \otimes h) \,.\, g) = \alpha(m \otimes hg) = mhg$$

and

$$\alpha(m \otimes h) \,.\, g = mh \,.\, g = mhg.$$

To see that α is a comodule map, we note that

$$(\alpha \otimes \mathrm{id})\rho(m \otimes h) = \sum (\alpha \otimes \mathrm{id})(m \otimes h_1 \otimes h_2) = \sum m h_1 \otimes h_2$$

and

$$\rho \circ \alpha(m \otimes h) = \rho(mh) = \sum m_0 h_1 \otimes m_1 h_2 = \sum m h_1 \otimes h_2,$$

since $\rho(m) = m \otimes 1$ because $m \in M^{\mathrm{coinv}}$.

Step 1–4 proves the claim and thus the theorem. □

cor:fundthmhopfmod **Corollary 3.39 (Fundamental Theorem of Hopf Modules).** *Every (right,right) Hopf module over a Hopf algebra is free as a Hopf module.*

Proof. We see by the theorem that

$$M^{\mathrm{coinv}} \otimes_k H \cong \bigoplus_{i \in I} H,$$

where I indexes a basis of M^{coinv}. □

Remark. The same result (as the corollary) holds true for (left,left), (left,right) and (right,left) type Hopf modules. The theorem holds for these if S is bijective. △

Question: Given an algebra A, how do we know if A can be given a Hopf algebra structure?

Answer: This is unknown in general, but there are restrictions on A. ♣

E.g. we will show the following result later.

Theorem. *Let H be a finite dimensional Hopf algebra. Then H is a* Frobenius algebra.[3]

Suppose M is a left A-module, where A is a k-algebra. Then $M^* = \mathrm{Hom}_k(M, k)$ is a right A-module (since M is a (A, k)-bimodule), where for $f \in M^*, a \in A$,

$$[fa](m) = f(am).$$

Similarly, if M is right A-module, then M^* is a left A-module with

$$[af](m) = f(ma),$$

and if M is an (A, A)-bimodule, then M^* is also an (A, A)-bimodule.

In particular A^* is an (A, A)-bimodule.

`thm:Frobalg` **Theorem 3.40.** *Let A be a finite dimensional k-algebra. Then the following are equivalent:*

(1) $A \cong A^*$ as right A-modules.

(1') $A \cong A^*$ as left A-modules.

(2) There is a nondegenerate bilinear form $(\cdot, \cdot)$ on A (i.e. $(\cdot, \cdot)\colon A \times A \to k$) such that the form is associative, *i.e. $(ab, c) = (a, bc)$.*

(3) There is a linear functional $f\colon A \to k$ such that $\mathrm{Ker}\, f$ contains no nonzero right ideals of A.

(3') There is a linear functional $f\colon A \to k$ such that $\mathrm{Ker}\, f$ contains no nonzero left ideals of A.

`def:Frobalg` **Definition 3.41 (Frobenius algebra).** We say an algebra A is *Frobenius* if it satisfies any of the conditions from Theorem 3.40.

[3] We will define this shortly.

Proof (of theorem). **(1) $\implies$ (2):** Let $\phi A \to A^*$ be an isomorphism of right modules. Define a form $(\,\cdot\,,\,\cdot\,)$ by

$$(a, b) = \phi(a)(b).$$

Recall that $(\,\cdot\,,\,\cdot\,)$ is nondegenerate if there does not exist $0 \neq a \in A$ such that $(a, b) = 0$ for all $b \in A$. Note $\phi(a) = (a,\,\cdot\,)$, so if $(a,\,\cdot\,) = 0$, then $\phi(a) = 0$, and thus $a = 0$ since ϕ is bijective. So $(\,\cdot\,,\,\cdot\,)$ is nondegenerate.

Now

$$(ab, c) = \phi(ab)(c) = [\phi(a)b](c) = \phi(a)(bc) = (a, bc).$$

(2) $\implies$ (3): Assume we have a form $(\,\cdot\,,\,\cdot\,)$, and consider $f = (1_A,\,\cdot\,)\colon A \to k$. If $\operatorname{Ker} f$ contains a nonzero right ideal, then it contains aA for some $a \neq 0$. So

$$f(aA) = (1_A, aA) = (a, A) = 0,$$

but $(\,\cdot\,,\,\cdot\,)$ is nondegenerate – contradiction!

(3) $\implies$ (1): Let f be such a linear function and define

$$\begin{aligned} \phi\colon A &\to A^* \\ 1 &\mapsto f \\ a &\mapsto fa, \end{aligned}$$

which is a right A-module map. If $\phi(a) = 0$, then $fa = 0$, so $f(ab) = [fa](b) = 0$ for all b, and thus $f(aA) = 0$. Hence $a = 0$, and thus ϕ is injective.

Now, since $\dim_k A < \infty$,

$$\dim_k A^* = \dim_k A < \infty,$$

so ϕ is an isomorphism.

Finally (1') $\implies$ (2) $\implies$ (3') $\implies$ (1') is similar. □

ex:MnkFrob **Example 3.42.** Consider the algebra $A = M_n(k)$. We claim that A is Frobenius. To see this, define a form on A with

$$(e_{ij}, e_{st}) = \delta_{sj}\delta it = \begin{cases} 1 & \text{if } e_{st} = e_{ji}, \\ 0 & \text{otherwise.} \end{cases}$$

This is the same form as

$$(P, Q) = \operatorname{tr}(PQ)$$

for $P, Q \in A = M_n(k)$. So $(\,\cdot\,,\,\cdot\,)$ is associative since

$$(P, QR) = \operatorname{tr}(PQR) = (PQ, R).$$

The form is nondegenerate since if $P \neq 0$, we can write $P = \sum a_{ij}e_{ij}$ where at least one $a_{ij} \neq 0$, so

$$(P, e_{ji}) = a_{ij} \neq 0.$$

Hence $A = M_n(k)$ is Frobenius by Definition 3.41. ○

Example 3.43. Consider the algebra $A = k[x]/(x^n)$ for some $n \geq 1$. A is local (i.e. has a unique maximal ideal) with maximal ideal (x) and unique minimal ideal (x^{n-1}). To satisfy (3) of Definition 3.41, we just need a $f\colon A \to k$ such that

$$\operatorname{Ker} f \cap (x^{n-1}) = 0,$$

which we can get by choosing an f with $f(x^{n-1}) \neq 0$. ○

Example 3.44. Consider the algebra $A = k[x, y]/(x^2, xy, y^2)$, and note that A is a 3 dimensional algebra with basis $1, x, y$. We claim that A is *not* Frobenius. To see this, note that every k-subspace of the 2 dimensional ideal

$$\frac{(x, y)}{(x^2, xy, y^2)}$$

is an ideal of A. So if $f\colon A \to k$ is linear,

$$\operatorname{Ker} f \cap \frac{(x, y)}{(x^2, xy, y^2)}$$

is nonzero, since any two dimension 2 subspaces of A intersect. So condition (3) of Definition 3.41 fails to hold for all $f\colon A \to k$. ○

thm:hopfalgfrob **Theorem 3.45.** *Let H be a finite dimensional Hopf algebra. Then H is a Frobenius algebra.*

Before beginning the proof we will introduce some notation. Let H be a Hopf algebra. Then H^* is a left and right H-module, and we will use the notation

$$(h \rightharpoonup f)(a) = f(ah) = \langle f, ah\rangle$$

for the left action of $h \in H$ on $f \in H^*$ (applied to $a \in H$) and

$$(f \leftharpoonup h)(a) = f(ha) = \langle f, ha\rangle$$

for the right action of h on f (applied to a). We also have another left action (with a the following notation)

$$(h \rightharpoondown f)(a) = f(S(h)a) = \langle f, S(h)a\rangle = (f \leftharpoonup S(h))(a)$$

and a right action (with the following notation)

$$(f \leftharpoondown h)(a) = f(aS(h)) = \langle f, aS(h)\rangle = (S(h) \rightharpoonup f)(a).$$

Since H^* is a left H^*-module by multiplication, H^* is also a right H-comodule with

$$\begin{aligned} \rho\colon H^* &\to H^* \otimes H. \\ f &\mapsto \sum f_0 \otimes f_1. \end{aligned}$$

Recall, if

$$\begin{aligned} m\colon H^* \otimes H^* &\to H^*, \\ f \otimes g &\mapsto fg \end{aligned}$$

then

$$\begin{aligned} m\colon H^* \otimes H^* \xrightarrow{\operatorname{id}\otimes\rho} H^* \otimes H^* \otimes H &\longrightarrow H^*, \\ f \otimes g \otimes a &\longmapsto \langle f, a\rangle g \end{aligned}$$

so

$$fg = \sum \langle f, g_1\rangle g_0. \tag{3.1}$$ {eq:fgH*}

Now (H^*, ρ) is a right H-comodule and $(H^*, \leftharpoonup)$ is a right H-module.

Proof (of Theorem 3.45). We will split the proof into several steps.

Step 1: H^* is a (right,right) Hopf module under these structures.

We will show this by proving that ρ is a H-module map, so we need to show that

$$\sum (f_0 \leftharpoonup h_1) \otimes f_1 h_2 = \rho(f \leftharpoonup h) = \rho(f) \,.\, h = \Big(\sum f_0 \otimes f_1\Big) \,.\, h. \tag{3.2}$$ {eq:rhoactcom}

Given $g \in H^*$,

$$g(f \leftharpoonup h) = \sum \langle g, (f \leftharpoonup h)_1\rangle (f \leftharpoonup h)_0$$

by eq. (3.1). To show eq. (3.2) it is enough to show that if we apply $\operatorname{id}_{H^*} \otimes g$ to both sides, we get equal results for all $g \in H^*$. So to show eq. (3.2) it is enough to show that

$$\begin{aligned} \sum \langle g, f_1 h_2\rangle (f_0 \leftharpoonup h_1) &= (\operatorname{id}\otimes g)\big(\rho(f \leftharpoonup h)\big) \\ &= \sum (\operatorname{id}\otimes g)\big((f \leftharpoonup h)_0 \otimes (f \leftharpoonup h)_1\big) \\ &= \langle g, (f \leftharpoonup h)_1\rangle (f \leftharpoonup h)_0 \\ &= g(f \leftharpoonup h), \end{aligned}$$

i.e. it is enough to show that

$$g(f \leftharpoonup h) = \sum \langle g, f_1 h_2\rangle (f_0 \leftharpoonup h_1) \tag{3.3}$$ {eq:gfSracth}

for all $f, g \in H^*, h \in H$. To show this we start with the right hand side:

$$\begin{aligned} \sum \langle g, f_1 h_2\rangle (f_0 \leftharpoonup h_1) &= \sum \langle h_2 \rightharpoonup g, f_1\rangle (f_0 \leftharpoonup h_1) \\ &= \sum \big(\langle h_2 \rightharpoonup g, f_1\rangle f_0\big) \leftharpoonup h_1 \\ &= \sum \big[(h_2 \rightharpoonup g) f\big] \leftharpoonup h_1 \end{aligned}$$

by eq. (3.1). Now it is enough to show for all $x \in H$ that both sides of eq. (3.3) are the same. We see that

$$
\begin{aligned}
&\sum \big\langle [(h_2 \rightharpoonup g)f] \leftharpoondown h_1, x \big\rangle \\
&\quad = \sum \big\langle [(h_2 \rightharpoonup g)f], xS(h_1) \big\rangle \\
&\quad = \sum \big\langle h_2 \rightharpoonup g, (xS(h_1))_1 \big\rangle \big\langle f, (xS(h_1))_2 \big\rangle && \text{(by def. of mult. in } H^*\text{)} \\
&\quad = \sum \big\langle h_2 \rightharpoonup g, x_1 S(h_1)_1 \big\rangle \big\langle f, x_2 S(h_1)_2 \big\rangle \\
&\quad = \sum \big\langle h_3 \rightharpoonup g, x_1 S(h_2) \big\rangle \big\langle f, x_2 S(h_1) \big\rangle && \text{(since } S \text{ anti-coalgebra map)} \\
&\quad = \sum \big\langle g, x_1 S(h_2) h_3 \big\rangle \big\langle f, x_2 S(h_1) \big\rangle \\
&\quad = \sum \big\langle g, x_1 \varepsilon(h_2) \big\rangle \big\langle f, x_2 S(h_1) \big\rangle && \text{(by axiom for} S\text{)} \\
&\quad = \sum \big\langle g, x_1 \big\rangle \big\langle f, x_2 S(h_1 \varepsilon(h_2)) \big\rangle \\
&\quad = \sum \big\langle g, x_1 \big\rangle \big\langle f, x_2 S(h) \big\rangle \\
&\quad = \sum \big\langle g, x_1 \big\rangle \big\langle f \leftharpoondown h, x_2 \big\rangle \\
&\quad = \big\langle g(f \leftharpoondown h), x \big\rangle,
\end{aligned}
$$

as we wanted.

Step 2: $(H^*, \leftharpoondown) \cong (H^*, \,.\,)$ as right H-modules.

Now we know that H^* is a Hopf module, so by the Fundamental Theorem of Hopf modules (Corollary 3.39)

$$H^* \cong (H^*)^{\mathrm{coinv}} \otimes_k H.$$

Furthermore, H is finite dimensional, so $\dim_k H^* = \dim_k H$, and thus $\dim_k (H^*)^{\mathrm{coinv}} = 1$ by the above, and hence

$$H^* \cong H$$

as right Hopf modules. But this only shows that

$$(H^*, \leftharpoondown) \cong (H, \,.\,)$$

as right H-modules, and to show H is Frobenius we want

$$(H^*, \leftharpoonup) \cong (H, \,.\,)$$

as right H-modules.

Step 3: S is bijective.

Choose an isomorphism of right H-modules

$$\begin{aligned}\phi\colon H &\to (H^*, \leftharpoondown).\\ 1 &\mapsto f\\ h &\mapsto (f \leftharpoondown h)\end{aligned}$$

If $S(h) = 0$ for $h \in H$, then

$$\langle f \leftharpoondown h, x\rangle = \langle f, xS(h)\rangle = 0$$

for all $x \in H$, so $f \leftharpoondown h = 0$, and thus $\phi(h) = 0$. Now, since ϕ is an isomorphism, we have that $h = 0$, and thus S is injective.

Since $\dim_k H = \dim_k H^*$ and S is linear, S is bijective.

Step 4: The map

$$\begin{aligned}\psi\colon (H^*, \leftharpoondown) &\to (H^*, \leftharpoonup)\\ f &\mapsto f \circ S\end{aligned}$$

is a right H-module isomorphism.

We note that ψ is bijection since S is (so $\psi^{-1} = [f \mapsto f \circ S^{-1}]$). Finally,

$$\begin{aligned}\big\langle \psi(f \leftharpoondown h), x\big\rangle &= \big\langle (f \leftharpoondown h) \circ S, x\big\rangle\\ &= \big\langle f \leftharpoondown h, S(x)\big\rangle\\ &= \big\langle f, S(h)S(x)\big\rangle\\ &= f\big(S(x)S(h)\big)\\ &= f \circ S(hx)\\ &= \big\langle (f \circ S) \leftharpoonup h, x\big\rangle\\ &= \big\langle \psi(f) \leftharpoonup h, x\big\rangle,\end{aligned}$$

so ψ is a right H-module map.

Step 5: $(H, .\,) \cong (H^*, \leftharpoonup)$.

Step 2 and Step 4 together implies that

$$(H^*, \leftharpoonup) \cong (H^*, \leftharpoondown) \cong (H, .\,)$$

as right H-modules, and thus H is Frobenius. $\square$

Corollary 3.46. *If H is a finite dimensional Hopf algebra, then S is bijective, and thus an anti-isomorphism.*

Proof. We proved this in Step 3 of the above proof. $\square$

Remark. Most nice infinite dimensional Hopf algebras also satisfy the above corollary. △

Remark. A finite dimensional Frobenius algebra A need not have a Hopf algebra structure. △

Example 3.47. Consider the algebra $A = M_n(k)$ for $n \geq 2$. We saw in Example 3.42 that A is Frobenius. But A is simple (has no ideals other than 0 and A), so if A is a bialgebra in some way, then $\operatorname{Ker}\varepsilon$ is an ideal with $\dim_k A/\operatorname{Ker}\varepsilon = 1$. Hence A cannot have a Hopf algebra structure (it can't even have a bialgebra structure). ○

Example 3.48. Let G be a finite group. Then kG is Frobenius. In fact kG has a Frobenius form where for $g, h \in G$,

$$(g,h) = \begin{cases} 1 & \text{if } gh = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

○

3.7 Integrals

Definition 3.49. Let H be a bialgebra. A *left integral* in H is a $t \in H$ such that

$$ht = \varepsilon(h)t$$

for all $h \in H$, and a *right integral* is a $t \in H$ such that

$$th = \varepsilon(h)t$$

for all $h \in H$. We write

$$\int_H^l = \{t \in H \mid t \text{ is a left integral}\}$$

and

$$\int_H^r = \{t \in H \mid t \text{ is a right integral}\}.$$

Note that both $\int_H^l$ and $\int_H^r$ are k-subspaces of H.

From the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\varepsilon) \longrightarrow H \xrightarrow{\varepsilon} k \longrightarrow 0$$

we get see that

$$k \cong H/\operatorname{Ker}(\varepsilon).$$

Actually $\operatorname{Ker}\varepsilon$ is an ideal, so k a an (H,H)-bimodule. k is called the *trivial module.*

If $\lambda \in k$, k the trivial module, and $h \in H$, then $h - \varepsilon(h) \in \operatorname{Ker}\varepsilon$ since $(h - \varepsilon(h)) \,.\, \lambda = 0$, i.e.

$$h \,.\, \lambda = \varepsilon(h)\lambda,$$

and similarly

$$\lambda \,.\, h = \varepsilon(h)\lambda.$$

We see that $0 \neq t \in H$ is a left integral if and only if kt is a left ideal of H and ${}_H(kt) \cong_H k$ (as left H-modules).Similarly $0 \neq t \in H$ is a right integral if and only if kt is a right ideal and $(kt)_H \cong k_H$ (as right H-modules).

Proposition 3.50. *Let H be a finite dimensional Hopf algebra. Then:*

(1) $\int_H^l$ and $\int_H^r$ are 1 dimensional.

(2) $S(\int_H^l) = \int_H^r$ and $S(\int_H^r) = \int_H^l$.

Proof. Consider H^* which is again a Hopf algebra (since H is finite dimensional). A left integral in H^* is $f \in H^*$ such that

$$gf = \varepsilon_{H^*}(g)f = (u_H)^*(g)f = g(1_H)f.$$

Recall from the proof of Theorem 3.45 that (H^*, ρ) is a right H-comodule, where

$$\begin{aligned} \rho\colon H^* &\to H^* \otimes H, \\ f &\mapsto \sum f_0 \otimes f_1 \end{aligned}$$

satisfies

$$gf = \sum \langle g, f_1 \rangle f_0 \tag{3.4}$$ {eq:gfinsum}

for all $f, g \in H^*$. From this we see $f \in \int_H^l$ if and only if

$$\langle g, 1 \rangle f = g(1)f = gf = \sum \langle g, f_1 \rangle f_0$$

for all $g \in H^*$. This forces

$$\rho(f) = \sum f_0 \otimes f_1 = f \otimes 1,$$

or equivalently $f \in (H^*)^{\mathrm{coinv}}$.

In the proof of Theorem 3.45 we also saw that $\dim_k (H^*)^{\mathrm{coinv}} = 1$, so $\dim_k \int_{H^*}^l = 1$. Similarly, $\dim_k \int_{H^*}^r = 1$.

This proves (1) since as H runs over all finite dimensional Hopf algebras, so does H^* (since $H^{**} \cong H$ in this case).

For (2), let $t \in \int_H^l$ so that $ht = \varepsilon(h)t$ for all $h \in H$. Then

$$\begin{aligned} S(t)h &= S(t)SS^{-1}(h) \\ &= S\big(S^{-1}(h)t\big) && (S \text{ is an anti-isomorphism}) \\ &= S\big(\varepsilon(S^{-1}(h))t\big) \\ &= \varepsilon S^{-1}(h)S(t) \\ &= \varepsilon(h)S(t) \end{aligned}$$

where S is bijective by the proof of Theorem 3.45, and where $\varepsilon \circ S^{-1} = \varepsilon$ is equivalent to $\varepsilon = \varepsilon \circ S$, which is true since S is an anti-homomorphism of coalgebras. So $S(t) \in \int_H^r$. Similarly, if $t \in \int_H^r$, then $S(t) \in \int_H^l$.

Since S is bijective and $\dim_k \int_H^l = \dim_k \int_H^r = 1$, we see that $S(\int_H^l) = \int_H^r$ and $S(\int_H^r) = \int_H^l$. □

Example 3.51. Let G be a finite group and $H = kG$. Then $t = \sum_{g\in G} g$ is a left and right integral, so $kt = \int_H^l = \int_H^r$. To see this, note that

$$g't = g' \sum_{g\in G} g \sum_{g\in G} g'g = \sum_{h\in G} h = t,$$

and $\varepsilon(g') = 1$ for all $g' \in G$. So $t \in \int_H^l$ and a similar argument shows that $t \in \int_H^r$. ○

Example 3.52. Let G be a finite group and $H = (kG)^*$. Write P_g for the element g^* of the dual basis to G, so $(P_g)(h) = \delta_{gh}$. We claim that P_{1_G} is a left and right integral. To see this, note that

$$P_g P_1(h) = P_g(h)P_1(h) = \delta_{gh}\delta_{1h} = \delta_{g1}\delta 1h = P_g(1)P_1(h),$$

so $P_g P_1 = P_g(1)P_1$, where $\varepsilon_H(P_g) = P_g(1)$. ○

Definition 3.53. A finite dimensional Hopf algebra is *unimodular* if

$$\int_H^l = \int_H^r.$$

So by the above kG and $(kG)^*$, for a finite group G, are unimodular.

Example 3.54. Consider the 4 dimensional Taft algebra

$$H = \frac{k\langle x, y\rangle}{(x^2 - 1, y^2, xy + yx)} = k + kx + ky + kxy$$

with

$$\begin{aligned} &\Delta(x) = x \otimes x && \varepsilon(x) = 1 && S(x) = x \\ &\Delta(y) = y \otimes 1 + x \otimes y && \varepsilon(y) = 0 && S(y) = -xy. \end{aligned}$$

We claim that H is *not* unimodular. To see this, note that $y+xy = (1+x)y \in \int_H^l$ since

$$\begin{aligned}
1 \cdot (1+x)y &= \varepsilon(1)(1+x)y, \\
x \cdot (1+x)y &= (x+x^2)y = (x+1)y = \varepsilon(x)(1+x)y, \\
y \cdot (1+x)y &= (y+yx)y = y^2 - xy^2 = 0 = \varepsilon(y)(1+x)y, \\
xy \cdot (1+x)y &= xy^2 + xyxy = 0 - x^2y^2 = 0 = \varepsilon(xy)(1+x)y,
\end{aligned}$$

where $\varepsilon(xy) = \varepsilon(x)\varepsilon(y) = 0$ by the definition of ε on a product. But on the other hand

$$y - xy = y + yx = y(1+x) \in \int_H^r,$$

so H cannot be unimodular (since $x + yx$ and $x - yx$ are linearly independent).

Note that

$$\begin{aligned}
S\big((1+x)y\big) &= S(y)S(1+x) = (-xy)(1+x) = -xy - xyx \\
&= -xy + x^2y = -xy + y = y(1+x) =\in \int_H^r. \qquad \bigcirc
\end{aligned}$$

lem:infdimnoideals **Lemma 3.55.** *Let H be an infinite dimensional Hopf algebra. Then H has no nonzero finite dimensional left or right ideals.*

Proof. Let I be a finite dimensional right ideal and let (H, Δ) be the standard right H-comodule on H. Then H is a rational left H^*-module with

$$f \star h = \sum \langle f, h_2 \rangle h_1$$

for $f \in H^*, h \in H$. So H is a left H^*-module, and a right H-module, but *not* a (H^*, H)-bimodule. These two actions on H, $. = \cdot$ of H and $\star$ of H^*, satisfy

$$(f \star k) \cdot k = \sum \big(S(k_2) \rightharpoonup f\big) \star (h \cdot k_1).$$

To see this, note that

$$\begin{aligned}
&\sum \big(S(k_2) \rightharpoonup f\big) \star (h \cdot k_1) \\
&\quad = \sum \big\langle S(k_3) \rightharpoonup f, h_2k_2 \big\rangle h_1k_1 \\
&\quad = \sum \big\langle f, h_2k_2S(k_3) \big\rangle h_1k_1 \\
&\quad = \sum \langle f, h_2 \rangle h_1k_1\varepsilon(k_2) \\
&\quad = \sum \langle f, h_2 \rangle h_1k \\
&\quad = (f \star h) \cdot k.
\end{aligned}$$

Now take $J = H^* \star I$, which is still finite dimensional since H is a rational left H^*-module and thus locally finite. So J is a right coideal of H by the correspondence between left H^* submodules of H and right subcomodules of H. Also, J is still a right ideal since

$$J \cdot H = (H^* \star I) \cdot H \subseteq H^* \star (I \cdot H) = H^* \star I = J.$$

Now J is a Hopf submodule of the right Hopf module H, so J is free the Fundamental Theorem of Hopf Modules (Corollary 3.39). But $\dim_k H = \infty$ and $\dim_k J < \infty$, so $J = 0$. Hence $I = 0$. Similarly H has no nonzero finite dimensional left ideals. $\square$

Corollary 3.56. *An infinite dimensional Hopf algebra H has no nonzero left or right integrals.*

Proof. If $t \in \int_H^r$, then kt is a right ideal of H. So by the Lemma 3.55, $t = 0$. Similarly on the left. $\square$

Remark. If H is a Hopf algebra, then so is $H^{\mathrm{op,cop}}$, which is an algebra with the opposite multiplication and comultiplication, i.e.

$$g * h = hg,$$
$$\Delta(g) = \sum g_2 \otimes g_1.$$

Similarly, H^{op} (opposite multiplication, same comultiplication) and H^{cop} (same multiplication, opposite comultiplication) are Hopf algebras as long as S is bijective (so in particular if H is finite dimensional). $\triangle$

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List of Symbols

Algebras and Coalgebras

(A, m, u)	the algebra A	4
C^*	$= \operatorname{Hom}_k(C, k)$ the dual of the coalgebra C (which is an algebra)	11
$c_{(1)} \otimes c_{(2)}$	(simplified) Sweedler notation for $\Delta(c)$, where $c \in C$ for some coalgebra C	9
(C, Δ, ε)	the coalgebra C	5
C/I	factor coalgebra for a coideal I in the coalgebra C	16
$\Delta \colon C \to C \otimes_k C$	the comultiplication	4
$\Delta^{(2)} \colon C \to C \otimes C \otimes C$	the map $\Delta^{(2)}(c) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$	8
δ_{ij}	the Kronecker delta	6
$\varepsilon \colon C \to k$	the counit	4
$\langle \cdot\, , \cdot \rangle \colon V^* \times V \to k$	the bilinear map $(f, v) \mapsto f(v)$ (we also have a similar bilinear map $V^* \otimes V^* \times V \otimes V \to k$)	17
k	a field	1
$m \colon A \otimes_k A \to A$	the multiplication map	3
$\phi^* \colon W^* \to V^*$	the pullback of the linear map $\phi \colon V \to W$	10
$\sum c_{(1)} \otimes c_{(2)}$	Sweedler notation for $\Delta(c)$, where $c \in C$ for some coalgebra C	8
$\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$	$= \sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = \Delta^{(2)}(c)$	9
$u \colon k \to A$	the unit map	3
V^*	the dual space of the k-vector space V	9

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