Math 207a Winter 2020 Homework 3

1. Give examples of bialgebras A which show that, unlike the case of Hopf algebras, the dimensions of the spaces of left and right integrals are not tightly controlled. In particular:

(a). There are bialgebras A where $\int_A^\ell$ has any desired vector space dimension, including 0 and ∞. In particular, an infinite dimensional bialgebra can have a nonzero left integrals.

(c). The dimensions of $\int_A^\ell$ and $\int_A^r$ can be different, and can differ by an arbitrary number.

(Hint: consider monoid algebras).

2. Let H and H' be finite dimensional Hopf algebras.

(a). Find the spaces of left and right integrals for the Hopf algebra $H \otimes H'$, in terms of the integrals of H and H'.

(b). Show that $H \otimes H'$ is semisimple if and only if H and H' are.

(c). Show that for general finite dimensional semsimple algebras A and A' over an arbitrary field k, $A \otimes_k A'$ might fail to be semisimple.

3. Show that a finite dimensional semisimple Hopf algebra H is unimodular (that is, $\int_H^\ell = \int_H^r$.)

4. Let H be a Hopf algebra. Recall that H^{op} is H with opposite multiplication but the same comultiplication, and H^{cop} is H with the opposite comultiplication but the same multiplication. $H^{\text{op,cop}}$ has both multiplication and comultiplication reversed.

(a). Sketch the proof that all of these variations are Hopf algebras, where S^{-1} is the antipode of H^{op} and H^{cop} and S is the antipode of $H^{\text{op,cop}}$.

(b). If H is the 4-dimensional Taft algebra, show that all of these variations are isomorphic to H itself as Hopf algebras, even though H is neither commutative nor cocommutative.

5. Let $H = k\langle g, h, y\rangle/(gh - 1, hg - 1, yg - \zeta gy)$ where $\zeta \in k$. Note that h is the inverse of g in H. We can also write this algebra informally as $k\langle g, g^{-1}, y\rangle/(yg - \zeta gy)$.

(a). Prove that H is a Hopf algebra, where g is grouplike and y is $(1, g)$-primitive.

(b). Find a formula for S and show that if ζ is not a root of unity, then S has infinite order.

6. Let A be a Frobenius algebra.

(a). For a given nondegenrate associative form $(-,-)$ on A, show there is a uniquely determined algebra automorphism $\mu : A \to A$, the *Nakayama automorphism*, such that $(a, b) = (b, \mu(a))$ for all $a, b \in A$.

(b). For automorphisms τ, σ of A, let ${}^{\tau}A^{\sigma}$ be the (A, A)-bimodule which is A as a vector space with left and right actions $x \star a \star y = \tau(x) a \sigma(y)$. Give A^* its standard (A, A)-bimodule structure. Show that as (A, A)-bimodules, $A^* \cong {}^{\mu^{-1}}A^1 \cong {}^1A^{\mu}$.

(c). If $\{-, -\}$ is a different nondegenerate associative form on A, leading to a different Nakayama automorphism μ', then $\mu' = i_x \circ \mu$ for some inner automorphism $i_x : a \mapsto xax^{-1}$ of A.

7. Let H be a Hopf algebra over a field k which is algebraically closed.

(a). Suppose that H is cocommutative and cosemisimple. Prove that H is isomorphic to a group algebra. (Hint: every simple cocommutative coalgebra is 1-dimensional).

(b). Let H be finite dimensional and suppose that H is commutative and semisimple. Then H is isomorphic to the dual of the group algebra of a finite group.

(c). Suppose that $\dim_k H = 3$. Show that if H is either semisimple or cosemisimple, then $H \cong k\mathbb{Z}_3$ as Hopf algebras. (In class we will eventually see how to use this to prove problem 3 on the previous homework, by showing that if $\operatorname{char} k = 0$ then H has to be semisimple and cosemisimple.)