Math 207a Winter 2020 Homework 2

1. Let A and B be bialgebras.

(a). Show that $A \otimes_k B$, with the usual tensor product algebra and coalgebra structures, is a bialgebra. If A and B are Hopf algebras, so is $A \otimes_k B$.

(b). Show that $A \oplus B$, with the usual direct sum algebra and coalgebra structures, is not a bialgebra.

2. Classify up to isomorphism all bialgebras B over the field k such that $\dim_k B = 2$. Which of these are Hopf algebras? (the final answer depends on the characteristic). (Hint: One way is to first show that as an algebra, B is isomorphic to either $k[x]/(x^2)$ or to $k[x]/(x(x-1))$. You can choose such an isomorphism for which the kernel of the counit ϵ is equal to (x). Then classify the possible coproducts.)

3. Let k be algebraically closed of characteristic 0. It is known that any Hopf algebra H with $\dim_k H = 3$ over such a k is isomorphic to the group algebra kG where $G = \mathbb{Z}/3\mathbb{Z}$. Can you prove it? (Note: I am not sure of a good argument that uses only techniques we have discussed so far, but I am curious to see if anyone can find one. More generally, it is true that any Hopf algebra of order p is isomorphic to $k\mathbb{Z}/p\mathbb{Z}$, when p is prime. This result was not proved early in the development of the subject, but only much more recently).

4. Let $C = k[S]$ be a grouplike coalgebra on the k-basis S.

(a). Classify all right comodules over C up to isomorphism.

(b). Recall that C^* is the pointwise algebra of functions $f : S \to k$. Describe the rational C^*-modules, and verify explicitly that rational left C^*-modules correspond to right C-comodules, as stated in lecture.

5. Let $C = M_n(k)$ be the $n \times n$ matrices as a coalgebra with $\epsilon(e_{ij}) = \delta_{ij}$ and $\Delta(e_{ij}) = \sum_\ell e_{i\ell} \otimes e_{\ell j}$, where the $\{e_{ij}\}$ are the matrix units.

Classify all right C-comodules up to isomorphism.

6. Let $C = k[x]$ as a coalgebra with $\Delta(x^n) = \sum_{0 \le i \le n} x^i \otimes x^{n-i}$ and $\epsilon(x^n) = \delta_{0n}$.

Prove that any right comodule (N, ρ) over C can be described by giving a vector space N, and a locally nilpotent linear transformation $\phi : N \to N$, such that $\rho(n) = \sum_{s \ge 0} \phi^s(n) \otimes x^s$. A sketch was given in class by classifying rational C^*-modules instead. You can either flesh out the details of that proof, or prove this result directly.

7. Let H be a Hopf algebra. Consider H as a Hopf module, with the usual right module and comodule structures over itself.

(a). Show that the space of coinvariants, that is $H^{\text{coinv}} = \{h \in H | \Delta(h) = h \otimes 1\}$, is equal to k.

(b). Suppose that $I \subseteq H$ is a right ideal and a right coideal. Then I is a Hopf submodule of H, that is, a subspace which is itself a Hopf module under the restricted module and comodule actions. Using the fundamental theorem of Hopf modules, show that $I = 0$ or $I = H$.

(c). Suppose that M be a monoid which is not a group, and let kM be the monoid bialgebra, which is the monoid algebra and grouplike coalgebra on the basis M. We have seen that kM is a bialgebra which is not a Hopf algebra. Show that the fundamental theorem of Hopf modules does not hold for kM, because kM has proper right-bi-ideals, contradicting part (b). This demonstrates that although Hopf modules can be defined for any bialgebra, their structure is only tightly controlled in the Hopf algebra case.

8. Suppose that H is a Hopf algebra and that N is a left H-module. Consider $H \otimes_k N$ as a left H-module with the usual tensor product structure $h \cdot (g \otimes n) = \sum h_{(1)} g \otimes h_{(2)} n$. Make $H \otimes_k N$ into a left H-comodule using the first tensor coordinate only, i.e. $\rho(g \otimes n) = \sum g_{(1)} \otimes g_{(2)} \otimes n$.

(a). Show that under these structures $H \otimes_k N$ is a (left, left) Hopf module. Conclude that the left H-module structure on $H \otimes_k N$ is free.

(b). Let P be any projective left H-module. Let N be any left H-module. Show that the tensor product module $P \otimes_k N$ is again a projective left H-module. (This result has many applications in the study of homological algebra over a Hopf algebra)