Math 207a Winter 2020 Homework 1

April 6, 2020

1. Let V and W be vector spaces over k. Consider the canonical map $\psi : V^* \otimes_k W^* \to (V \otimes_k W)^*$, where given $f \in V^*, g \in W^*$, $[\psi(f \otimes g)](v \otimes w) = f(v)g(w)$.

(a). Show that ψ is injective.

(b). Show that ψ is an isomorphism of vector spaces if and only if either V or W is finite dimensional over k.

2. Keep the hypotheses of exercise 1.

Show that for subspaces $I \subseteq V^*$ and $J \subseteq W^*$ we have

$$(I \otimes J)^\perp = I^\perp \otimes W + V \otimes J^\perp.$$

(The notation is defined in the lectures). Hint: choose bases.

3. Let (C, Δ, ϵ) be a coalgebra over the field k. A nonzero element $c \in C$ is called *grouplike* if $\Delta(c) = c \otimes c$.

(a). Show that if c is grouplike, then $\epsilon(c) = 1$.

(b). Show that any set of distinct grouplike elements in C is linearly independent over k.

4. Recall that a coalgebra C is *grouplike* if it has a k-basis of grouplike elements.

(a). Show that any cosubalgebra of a grouplike coalgebra is again grouplike.

(b). Show that if C and D are grouplike coalgebras, then so are $C \oplus D$ and $C \otimes_k D$.

5. Let (C, Δ, ϵ) be a coalgebra over the field k. Given grouplike elements $g, h \in C$, an element $c \in C$ is called (g, h)-*primitive* if $\Delta(c) = g \otimes c + c \otimes h$.

(a). Show that if c is (g, h)-primitive, then $\epsilon(c) = 0$.

(b). Let V be the set of (g, h)-primitive elements of V. Show that V is a subspace of C; $D = V + kg + kh$ is a cosubalgebra of C; V is a coideal of D; and D/V is a grouplike coalgebra.

6. Let $C = k\mathbb{N}$ be the monoid coalgebra of $\mathbb{N} = \{x^0, x^1, x^2, \dots\}$. So we can think of C as the polynomial ring $k[x]$ as a vector space, with coproduct $\Delta(x^n) = \sum_{i+j=n} x^i \otimes x^j$ and counit $\epsilon(x^n) = \delta_{0n} = \begin{cases} 1 & \text{n} = 0 \\ 0 & n \neq 0 \end{cases}$.

(a) Show that the dual algebra C^* is isomorphic to the power series ring $k[[x]]$ in one variable over k.

(b). Find all cosubalgebras of C.

7. Recall that for a coalgebra C we showed that its dual C^* is an algebra.

(a). Suppose that $\phi : C \to D$ is a homomorphism of coalgebras. Show that the dual map $\phi^* : D^* \to C^*$ is a homomorphism of algebras.

(b). Suppose that C and D are coalgebras and consider the map $\psi : C^* \otimes D^* \to (C \otimes D)^*$ from exercise 1. Show that ψ is a homomorphism of algebras.