MATH 207A (Hopf Algebras) Homework

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Notations and Conventions

- K is a field.
- V and W are K-vector spaces.
- Unadorned $\otimes$ symbols are always over K.

Homework 1

1.1 Problem 1

Proposition 1.1. *The natural map $\psi_{V,W} \colon V^* \otimes W^* \to (V \otimes W)^*$ is always injective, and it is surjective if and only if one of V or W is finite-dimensional.*

The first statement may be proven using bases, and this proof is reasonable. However, I prefer going through one of my favorite lemmas about the tensor product.

Definition 1.2. $\mathscr{S} \subseteq V^*$ is called **separating** if $f(v) = 0$ for every $f \in \mathscr{S}$ implies $v = 0$.

For two linear functionals $f \in V^*$, $g \in W^*$, abuse notation slightly by writing $f \otimes g$ also for $\psi_{V,W}(f \otimes g) \in (V \otimes W)^*$.

Lemma 1.3. *Let $\mathscr{S} \subseteq V^*$, $\mathscr{T} \subseteq W^*$ be separating sets. If $x \in V \otimes W$ is such that $(f \otimes g)(x) = 0$, for every $f \in \mathscr{S}$ and $g \in \mathscr{T}$, then $x = 0$.*

Proof. Write $x = \sum_{j=1}^m v_j \otimes w_j$, where $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_m \in W$, where we take $w_1, \ldots, w_m \in W$ linearly independent. (This is possible basically because finite-dimensional vector spaces have bases.) Now, the condition implies that if $f \in \mathscr{S}$, then

$$0 = (f \otimes g)(x) = \sum_{j=1}^m f(v_j) g(w_j) = g\left(\sum_{j=1}^m f(v_j) w_j\right),$$

for every $g \in \mathscr{T}$. Since $\mathscr{T}$ is separating, $\sum_{j=1}^m f(v_j) w_j = 0$. Since $w_1, \ldots, w_m \in W$ are linearly independent, it follows that $f(v_1) = \cdots = f(v_m) = 0$, for every $f \in \mathscr{S}$. Since $\mathscr{S}$ is separating, we get $v_1 = \cdots = v_m = 0$. Therefore, $x = 0$, as desired. $\square$

The key observation is that if $e_V : V \hookrightarrow V^{**}$ is the natural map, then $e_V(V) \subseteq V^{**}$ is a separating set in $(V^*)^*$. Indeed, if $f \in V^*$, then $e_V(v)(f) = f(v)$, for all $v \in V$.

Proof of Injectivity Part of Proposition 1.1. Suppose $x \in V^* \otimes W^*$ is such that $\psi_{V,W}(x) \equiv 0$. For every $v \in V$ and $w \in W$, note the functional $e_V(v) \otimes e_W(w) : V^* \otimes W^* \to K$ satisfies

$$(e_V(v) \otimes e_W(w))(x) = \psi_{V,W}(x)(v \otimes w) = 0.$$

By Lemma 1.3 and the observation above, we conclude $x = 0$, as desired. □

For the second part of Proposition 1.1, we make an observation.

Lemma 1.4. *Let $(v_n)_{n\in\mathbb{N}} \in V^{\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}} \in W^{\mathbb{N}}$ be sequences in V and W, respectively. If $h := \sum_{j=1}^m f_j \otimes g_j \in \operatorname{im} \psi_{V,W} \subseteq (V \otimes W)^*$, then*

$$\operatorname{rank} \begin{bmatrix} h(v_1 \otimes w_1) & \cdots & h(v_1 \otimes w_n) \\ \vdots & \ddots & \vdots \\ h(v_n \otimes w_1) & \cdots & h(v_n \otimes w_n) \end{bmatrix} \leq m,$$

for all $n \in \mathbb{N}$.

Proof. Note that for all $j, k \in [n]$, we have

$$h(v_j \otimes w_k) = \sum_{\ell=1}^m f_\ell(v_j)\, g_\ell(w_k) = (F_n G_n)_{jk},$$

where

$$F_n = \begin{bmatrix} f_1(v_1) & \cdots & f_m(v_1) \\ \vdots & \ddots & \vdots \\ f_1(v_n) & \cdots & f_m(v_n) \end{bmatrix} \in K^{n\times m} \text{ and } G_n = \begin{bmatrix} g_1(w_1) & \cdots & g_1(w_n) \\ \vdots & \ddots & \vdots \\ g_m(w_1) & \cdots & g_m(w_n) \end{bmatrix} \in K^{m\times n},$$

whence it follows that

$$\operatorname{rank} \begin{bmatrix} h(v_1 \otimes w_1) & \cdots & h(v_1 \otimes w_n) \\ \vdots & \ddots & \vdots \\ h(v_n \otimes w_1) & \cdots & h(v_n \otimes w_n) \end{bmatrix} = \operatorname{rank}(F_n G_n) \leq \operatorname{rank} F_n \leq m,$$

as desired. □

We are now ready for the rest of Proposition 1.1.

Proof of Second Part of Proposition 1.1. We first show $\psi_{V,W}$ is not surjective if V and W are both infinite-dimensional. In this case, there are sequence $(v_n)_{n\in\mathbb{N}} \in V^{\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}} \in W^{\mathbb{N}}$ of linearly independent vectors in both V and W. Then we know $(v_n \otimes w_n)_{n\in\mathbb{N}}$ is linearly independent in $V \otimes W$. Completing it to a basis (or choosing a complementary subspace), we conclude there exists $\varphi \in (V \otimes W)^*$ such that

$$\varphi(v_j \otimes w_k) = \delta_{jk},$$

for all $j, k \in \mathbb{N}$. But then $[\varphi(v_j \otimes w_k)]_{j,k\in[n]} = I_n \in K^{n\times n}$, for all $n \in \mathbb{N}$. Since $\operatorname{rank} I_n = n$, we conclude from Lemma 1.4 that $\varphi \notin \operatorname{im} \psi_{V,W}$, and therefore $\psi_{V,W}$ is not surjective.

Now, in the case that (without loss of generality) W is finite-dimensional, let $n \coloneqq \dim W$. Since all maps are natural, we may as well assume $W = K^n$. In this case, we can consider the identifications

$$\begin{aligned}(V^*)^n &\cong (V^* \otimes K)^n \\ &\cong (V^* \otimes K^*)^n \\ &\cong V^* \otimes (K^*)^n \\ &\cong V^* \otimes (K^n)^*.\end{aligned}$$

and

$$\begin{aligned}(V \otimes K^n)^* &\cong ((V \otimes K)^n)^* \\ &\cong (V^n)^* \cong (V^*)^n\end{aligned}$$

(These all work out because direct sums, i.e., coproducts, and direct products, i.e., products, of finitely many vector spaces are the same.) One may check easily that under these identifications,

$$(V^*)^n \cong V^* \otimes (K^n)^* \xrightarrow{\psi_{V,K^n}} (V \otimes K^n)^* \cong (V^*)^n$$

is the identity map, so that ψ_{V,K^n} is surjective. □

1.2 Problem 2

Proposition 1.5. *Let $I \subseteq V^*$ and $J \subseteq W^*$ be subspaces. Then $(I \otimes J)^\perp = I^\perp \otimes W + V \otimes J^\perp$.*

As above, there is a proof of this fact using bases, but I prefer to go through Lemma 1.3. Let V_1, V_2, W_1, W_2 all be K-vector spaces.

Lemma 1.6 (Flatness). *If $T_1 \colon V_1 \to W_1$ and $T_2 \colon V_2 \to W_2$ are injective linear maps, then $T_1 \otimes T_2 \colon V_1 \otimes V_2 \to W_1 \otimes W_2$ is injective as well.*

Proof. We first observe $\mathscr{S} \coloneqq \{f \circ T_1 \in V_1^* : f \in W_1^*\}$ and $\mathscr{T} \coloneqq \{g \circ T_2 \in V_2^* : g \in W_2^*\}$ are separating sets. Indeed, if $v \in V_1$ is such that $f(T_1 v) = 0$, for all $f \in W_1^*$, then $T_1 v = 0$ (because W_1^* is a separating set). Since T_1 is injective, we conclude $v = 0$, as desired. The same argument works for $\mathscr{T}$.

Now, suppose $x \in V_1 \otimes V_2$ is such that $(T_1 \otimes T_2)(x) = 0$. Then, for all $f \in W_1^*$ and $g \in W_2^*$, we have

$$((f \circ T_1) \otimes (g \circ T_2))(x) = ((f \otimes g) \circ (T_1 \otimes T_2))(x) = (f \otimes g)(0) = 0.$$

By the previous paragraph and Lemma 1.3, we conclude $x = 0$, as desired. □

Lemma 1.7 (Quotients). *If $V_1 \subseteq V$ and $W_1 \subseteq W$ are subspaces and $\pi_1 \colon V \to V/V_1$ and $\pi_2 \colon W \to W/W_1$ are the natural quotient maps, then $\ker(\pi_1 \otimes \pi_2) = V_1 \otimes W + V \otimes W_1$.*

Proof. We did this in class. □

Theorem 1.8. *If $T_1 : V_1 \to W_1$ and $T_2 : V_2 \to W_2$ are linear, then*

$$\ker(T_1 \otimes T_2) = \ker T_1 \otimes V_2 + V_1 \otimes \ker T_2.$$

Proof. Restricting the codomains of T_1 and T_2 to get $\tilde{T}_1 : V_1 \to \operatorname{im} T_1$ and $\tilde{T}_2 : V_2 \to \operatorname{im} T_2$, we get $T_1 \otimes T_2$ as the composition

$$V_1 \otimes V_2 \xrightarrow{\tilde{T}_1 \otimes \tilde{T}_2} \operatorname{im} T_1 \otimes \operatorname{im} T_2 \xrightarrow{\iota_1 \otimes \iota_2} W_1 \otimes W_2,$$

where $\iota_j : \operatorname{im} T_j \hookrightarrow W_j$ is inclusion, for $j \in \{1, 2\}$. Since $\iota_1 \otimes \iota_2$ is injective by Lemma 1.6,

$$\ker(T_1 \otimes T_2) = \ker(\tilde{T}_1 \otimes \tilde{T}_2) = \ker \tilde{T}_1 \otimes V_2 + V_1 \otimes \ker \tilde{T}_2 = \ker T_1 \otimes V_2 + V_1 \otimes \ker T_2,$$

by Lemma 1.7 and the First Isomorphism Theorem. □

Now, the key observation is that if $I \subseteq V^*$ is a subspace and $e_V^I := \rho_V^I \circ e_V : V \to I^*$ is the natural map $e_V : V \hookrightarrow V^{**}$ followed by the restriction map $\rho_V^I : (V^*)^* \to I^*$ defined by $f \mapsto f|_I$, then

$$I^\perp = \ker e_V^I,$$

by definition of $\perp$.

Proof of Proposition 1.5. Consider the map $e_V^I \otimes e_W^J : V \otimes W \to I^* \otimes J^*$. By Theorem 1.8, we have that

$$\ker(e_V^I \otimes e_W^J) = \ker e_V^I \otimes W + V \otimes \ker e_W^J = I^\perp \otimes W + V \otimes J^\perp.$$

But also, the universal property of the tensor product implies that $e_{V \otimes W}^{I \otimes J}$ is the composition

$$V \otimes W \xrightarrow{e_V^I \otimes e_W^J} I^* \otimes J^* \xrightarrow{\psi_{I,J}} (I \otimes J)^*$$

where $\psi_{I,J}$ is the injective map from Proposition 1.1. We conclude that

$$(I \otimes J)^\perp = \ker(e_{V \otimes W}^{I \otimes J}) = \ker(e_V^I \otimes e_W^J) = I^\perp \otimes W + V \otimes J^\perp,$$

as claimed. □

Math 207A HW 1 Problem 3

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1. Since C is a coalgebra, and $c \in C$ is group-like we have

$$\begin{aligned}(1 \otimes \epsilon) \circ \Delta(c) &= c \quad (1)\\ (1 \otimes \epsilon)(c \otimes c) &= c \quad (2)\\ c \cdot \epsilon(c) &= c \quad (3)\\ \epsilon(c) &= 1 \quad (4)\end{aligned}$$

(Note $c \neq 0$ since 0 is not a group-like element.) $\square$

2. Suppose the group-like elements are not linearly independent, then there is a linear depending relation

$$\alpha_1 c_1 + \cdots + \alpha_n c_n = 0,$$

where c_i, c are distinct group-like elements in C and α_i are elements in k for $i \in \{1, \cdots, n\}$. Assume $\{c_1, \cdots, c_n\}$ are the smallest possible set that has linear dependency, then we observe that $\alpha_i \neq 0$, $\forall i \in \{1, \cdots, n\}$ (Otherwise we would have a smaller linear dependent set). Furthermore, if $n = 1$ then we have

$$\alpha_1 c_1 = 0$$

which is a contradiction since 0 is not a group-like element in C, so $n > 1$.
Then we can write

$$c_n = \sum_{i=1}^{n-1} \beta_i c_i,$$

where $\beta_i \in k/\{0\}$ for all $i \in \{1, \cdots, n-1\}$. Since this is a smaller set of c_i's, it has to be linear independent.
Now we apply Δ to both sides of the equation above and we get

$$c_n \otimes c_n = \sum_{i=1}^{n-1} \beta_i c_i \otimes c_i.$$

Thus we get $1 = n - 1$ by looking at the rank of both sides, so we get $\beta_1 c_1 + \beta_2 c_2 = 0 \Rightarrow c_2 = \beta_1 c_1$. By applying ϵ to both sides we get $\epsilon(c_2) = \epsilon(\beta_1 c_1) = 1$, so $\beta_1 = 1 \Rightarrow c_1 = c_2$, which reaches a contradiction. $\square$

4. (a) Let $D \subseteq C$ be a subcoalgebra of a grouplike coalgebra. Let $\{g_i\}$ be a basis of grouplike elements in C. Let $d \in D$. Then $d = \sum_i a_i g_i$ for some $a_i \in k$. Then $\Delta(d) = \sum_i a_i g_i \otimes g_i$, but also $\Delta(d) \in D \otimes D \subseteq D \otimes C$ so $\Delta(d) = \sum_i v_i \otimes g_i$ for some $v_i \in D$. Since the g_i are linearly independent, we have that $a_i g_i = v_i \in D$ for all i. Therefore each $g_i \in D$ when $a_i \neq 0$. So d is in the span of grouplike elements. Hence all of D is the span of grouplike elements. Since the grouplike elements are linearly independent, D is grouplike.

(b) If C, D are grouplike coalgebras, then they have bases of grouplike elements $\{c_i\}$ and $\{d_j\}$ respectively.

A basis for $C \oplus D$ is given by $\{(c_i, 0), (0, d_j)\}$, and note that the coproduct on $C \oplus D$ is given by first applying $\Delta_C \oplus \Delta_D$ to $C \oplus D$ and then distributing the direct sum canonically. In other words, $(c, 0) \mapsto (c \otimes c, 0 \otimes 0) \mapsto (c, 0) \otimes (c, 0)$ and similarly for $(0, d)$ where $c \in C$ and $d \in D$ is grouplike. So the $(c_i, 0)$ and the $(0, d_j)$ are grouplike.

For the tensor product, $\{c_i \otimes d_j\}$ is a basis, and the coproduct is taken by first applying $\Delta_C \otimes \Delta_D$ to $C \otimes D$ and then applying τ_{23}. That is, $(c, d) \mapsto (c \otimes c) \otimes (d \otimes d) \mapsto (c \otimes d) \otimes (c \otimes d)$ and hence $c \otimes d$ is grouplike if c, d are.

So $C \oplus D$ and $C \otimes D$ are grouplike.

MATH 207A, HW 1, Problem 5

Cameron Cinel

April 17, 2020

Problem Statement

Let (C, Δ, ε) be a coalgebra over the field K. Given grouplike elements $g, h \in C$, and element $c \in C$ is called (g, h)-primitive if

$$\Delta(c) = g \otimes c + c \otimes h.$$

1. Show that if c is (g, h)-primitive, then $\varepsilon(c) = 0$.
2. Let V be the set of (g, h)-primitive elements of C. Show that:
   - V is a subspace of C;
   - $D = V + Kg + Kh$ is a subcoalgebra of C;
   - V is a coideal of D;
   - and D/V is a grouplike coalgebra.

$\varepsilon(c) = 0$

Recall that from Problem 3, we have that $\varepsilon(g) = 1$ for any grouplike $g \in C$.
Suppose $c \in C$ is (g, h)-primitive. Notice that

$$(\mathrm{id}_C \otimes \varepsilon)(\Delta(c)) = (\mathrm{id}_C \otimes \varepsilon)(g \otimes c + c \otimes h) = g \otimes \varepsilon(c) + c \otimes \varepsilon(h).$$

From the properties of coalgebras, it follows that

$$c = \varepsilon(c)g + \varepsilon(h)c = \varepsilon(c)g + c \iff \varepsilon(c)g = 0.$$

Since g is grouplike, $g \neq 0$ so $\varepsilon(c) = 0$. □

V is a subspace of C

Clearly $0 \in C$ is (g, h)-primitive, so we only have to show closure under addition and scalar multiplication. Suppose $c_1, c_2 \in V$ and $k \in K$. Then

$$\begin{aligned} \Delta(c_1 + c_2) &= \Delta(c_1) + \Delta(c_2) \\ &= g \otimes c_1 + c_1 \otimes h + g \otimes c_2 + c_2 \otimes h \\ &= g \otimes (c_1 + c_2) + (c_1 + c_2) \otimes h \end{aligned}$$

so $c_1 + c_2 \in V$. We also have that

$$\Delta(kc_1) = k\Delta(c_1) = k(g \otimes c_1 + c_1 \otimes h) = g \otimes (kc_1) + (kc_1) \otimes h$$

so $kc_1 \in V$. □

$D = V + Kg + Kh$ is a subcoalgebra of C

Recall that if $g \in G$ is grouplike, then $\Delta(g) = g \otimes g$.
Since D is a subspace of C, we only have to check that $\Delta(D) \subset D \otimes D$.
Suppose $d \in D$. Then we can write $d = c + k_1 g + k_2 h$ for $c \in V$ and $k_1, k_2 \in K$. Notice that

$$\Delta(d) = g \otimes c + c \otimes h + k_1(g \otimes g) + k_2(h \otimes h).$$

Since $g, h, c \in D$, we have that $\Delta(d) \in D \otimes D$.

V is a coideal of D

Recall that a subspace $I \subset C$ is a called a coideal of $\varepsilon(I) = 0$ and

$$\Delta(I) \subset I \otimes C + C \otimes I.$$

From part 1, we have that $\varepsilon(V) = 0$. Suppose $c \in V$. Then

$$\Delta(c) = c \otimes h + g \otimes c.$$

Since $g, h \in D$, we are done.

D/V is grouplike

To show this, we are going to show that $D \cong K$ as K-vector spaces. First we show that $V \cap (Kg + Kh) = K(g - h)$. Suppose that $c \in V \cap (Kg + Kh)$. Then we can write $c = k_1 g + k_2 h$. Therefore

$$0 = \varepsilon(c) = k_1\varepsilon(g) + k_2\varepsilon(h) = k_1 + k_2$$

so $k_2 = -k_1$ so $c \in K(g - h)$. Now notice that for any $k \in K$,

$$\begin{aligned}\Delta(kg - kh) =& g \otimes (kg) + (-kh) \otimes h \\ =& g \otimes (kg) - k(g \otimes h) + k(g \otimes h) + (-kh) \otimes h \\ =& g \otimes (kg - kh) + (kg - kh) \otimes h\end{aligned}$$

so $kg - kh \in V \cap (Kg + Kh)$.

D/V is grouplike

Consider the map

$$\phi : D \to K$$
$$c + k_1 g + k_2 h \mapsto k_1 + k_2.$$

First we show that ϕ is well-defined. Suppose

$$d_k := c_k + k_1 g + k_2 h = c_t + t_1 g + t_2 h =: d_t.$$

Then

$$(k_1 - t_1)g + (k_2 - t_2)h \in V \cap (Kg + Kh) = K(g - h).$$

Therefore

$$k_1 - t_1 + k_2 - t_2 = 0 \iff k_1 + k_2 = t_1 + t_2.$$

Thus

$$\phi(d_k) = k_1 + k_2 = t_1 + t_2 = \phi(d_t).$$

D/V is grouplike

Clearly ϕ is K-linear and surjective. Also since

$$\varepsilon(c + k_1 g + k_2 g) = k_1 + k_2 = \varepsilon(\phi(c + k_1 g + k_2 g))$$

and

$$(\phi \otimes \phi)(\Delta(c + k_1 g + k_2 h)) = k_1 \otimes 1 + k_2 \otimes 1 = \Delta(k_1 + k_2)$$

ϕ is a morphism of coalgebras. Thus we only have to show that $\ker \phi = V$. If $c \in V$, then $c = c + 0g + 0h$ so $\phi(c) = 0$.
Now suppose that $d \in \ker \phi$. We can write $d = c + k_1 g + k_2 h$ for $c \in V$ and $k_1, k_2 \in K$. Then $k_1 + k_2 = 0$ so $k_2 = -k_1$. Therefore $d \in V + K(g - h)$. Since $K(g - h) \subset V$, we have that $d \in V$ and so $\ker \phi = V$.
By the first isomorphism theorem, it follows that $D/V \cong K$. Thus any non-zero element forms a basis for D/V so $\{g + V\}$ is a K-basis of grouplike elements. □

6. Let $C = kN$ be the monoid coalgebra of $N = \{x^0, x^1, x^2, \dots\}$.

(a). Show that $C^* \cong k[[y]]$

(b). Find all subcoalgebra of C

Pf: (a) $\varphi : C^* \to k[[y]]$ WTS that φ is an algebra map

$f \mapsto \sum_{i=0}^{\infty} f(x^i) y^i$

$\Delta^*(f \otimes g)(x^n) = \sum_{i+j=n} f(x^i) \cdot g(x^j)$ = the coefficient of $\varphi(f)\varphi(g)$ on the degree n term

$\Rightarrow \varphi$ is an algebra homomorphism and is a bijection.

$\boxtimes$

(b). Define $k[n]$ as the subspace of kN with basis $\{x^0, x^1, \dots x^n\}$,

Claim: $k[n]$ are all the subcoalgebras of kN

Pf: V a vector space $W \subseteq V$, then $W \subseteq (W^{\perp})^{\perp}$ by definition

let $D \subseteq C$ be a subcoalgebra, then $D^{\perp} \trianglelefteq C^*$ is an ideal.

$k[[y]]$ is PID and (y^n) are the only ideal.

then $D^{\perp} = (y^n)$ for some n

and $(y^n)^{\perp} = k[n-1]$, for any $f \in (y^n)^{\perp}$, $f(x^i) = 0$ $\forall i < n$.

and suppose $\sum_{i=0}^{m} a_i x^i \in (y^n)^{\perp}$ with $m \geq n$,

then let $l = m-n$, we have $y^n \cdot y^l (\sum_{i=0}^{m} a_i x^i) = 1 \neq 0$

$\Rightarrow (y^n)^{\perp} = k[n-1]$

$k[n]$ is a subcoalgebra, $\Delta(x^m) = \sum_{i+j=m} x^i \otimes x^j \in k[n] \otimes k[n]$

WT find all the subcoalgebra of $k[n]$, which is f.d.

cosider $(k[n])^* \cong k[y]/(y^{n+1})$

$\{1, x, \dots x^n\}$ is a basis for $k[n]$

let $\{1', x', \dots x^{n'}\}$ be the dual basis

$\Delta^*(x^{i'} \otimes x^{j'}) = (x^{i+j})'$ then $(k[n])^*$ is a n-d k-algebra generated by x', and $(x')^{n+1} = 0$.

$\Rightarrow k[y]/(y^{n+1}) \cong (k[n])^*$ by correspondence theorem,

all ideals of $k[y]/(y^{n+1})$ look like (y^k) for nonnegative integer $k \leq n$

$(y^k)^\perp \cong k[k-1]$

∎

Problem 7.

Recall that for a coalgebra (C, Δ, ε), we have showed that the dual $(C^*, \Delta^*, \varepsilon^*)$ is an algebra.

(a) Suppose that $\phi\colon C \to D$ is a homomorphism of colagebras. Show that the dual map $\phi^*\colon D^* \to C^*$ is a homomorphism of algebras.

Solution: Note that ϕ being a homomorphism of coalgebras implies

$$(\phi \otimes \phi) \circ \Delta_C = \Delta_D \circ \phi \qquad \text{and} \qquad \varepsilon_C = \varepsilon_D \circ \phi.$$

Thus

$$\phi^*(1_{D^*}) = \phi^*(\varepsilon_D) = \varepsilon_D \circ \phi = \varepsilon_C = 1_{C^*}$$

and

$$\begin{aligned}\phi^*(fg) &= \phi^* \circ \Delta_D^*(f \otimes g) = (\Delta_D \circ \phi)^*(f \otimes g) = \big((\phi \otimes \phi) \circ \Delta_C\big)^*(f \otimes g)\\ &\overset{(*)}{=} \Delta_C^*(\phi^* f \otimes \phi^* g) = \phi^*(f)\phi^*(g)\end{aligned}$$

for $f \in C^*, g \in D^*$. Here $(*)$ is true since

$$\begin{aligned}\big[\big((\phi \otimes \phi) \circ \Delta_C\big)^*(f \otimes g)\big](c) &= (f \otimes g)\Big(\sum \phi(c_{(1)}) \otimes \phi(c_{(2)})\Big)\\ &= \sum f(\phi c_{(1)}) \otimes g(\phi c_{(2)})\\ &= \sum \phi^* f(c_{(1)}) \otimes \phi^* g(c_{(2)})\\ &= (\phi^* f \otimes \phi^* g) \circ \Delta_C(c)\\ &= \big[\Delta_C^* \circ (\phi^* f \otimes \phi^* g)\big](c)\end{aligned}$$

for $f \in C^*, g \in D^*, c \in C$. Hence ϕ^* is a homomorphism of algebras.

(b) Suppose that C and D are coalgebras and consider the map $\psi\colon C^* \otimes D^* \to (C \otimes D)^*$ given by $[\psi(f \otimes g)](v \otimes w) = f(v)g(w)$ for $f \in V^*, g \in W^*, v \in V, w \in W$. Show that ψ is a homomorphism of algebras.

Solution: Note that

$$\psi(1_{C^* \otimes D^*}) = \psi(\varepsilon_C \otimes \varepsilon_D) = \varepsilon_{C \otimes D} = 1_{(C \otimes D)^*}$$

since $\varepsilon_{C \otimes D}(c \otimes d) = \varepsilon_C(c)\varepsilon_D(d)$. Now note that for $f, g \in C^*, \alpha, \beta \in D^*$,

$$m_{C^* \otimes D^*} = (m_{C^*} \otimes m_{D^*}) \circ \tau_{23} = (\Delta_C^* \otimes \Delta_D^*) \circ \tau_{23}$$

implies that

$$\begin{aligned}\psi\big((f \otimes \alpha)(g \otimes \beta)\big) &= \psi \circ m_{C^* \otimes D^*}\big((f \otimes \alpha) \otimes (g \otimes \beta)\big)\\ &= \psi \circ (\Delta_C^* \otimes \Delta_D^*) \circ \tau_{23}\big((f \otimes \alpha) \otimes (g \otimes \beta)\big)\\ &= \psi \circ (\Delta_C^* \otimes \Delta_D^*)\big((f \otimes g) \otimes (\alpha \otimes \beta)\big)\\ &= \psi\big(\Delta_C^*(f \otimes g) \otimes \Delta_D^*(\alpha \otimes \beta)\big), \qquad (1)\end{aligned}$$

and

$$m_{(C\otimes D)^*} = \Delta^*_{C\otimes D} = \big(\tau_{23}\circ(\Delta_C\otimes\Delta_D)\big)^*$$

implies that

$$\begin{aligned}\psi(f\otimes\alpha)\psi(g\otimes\beta) &= m_{(C\otimes D)^*}\big(\psi(f\otimes\alpha)\otimes\psi(g\otimes\beta)\big)\\ &= \big(\tau_{23}\circ(\Delta_C\otimes\Delta_D)\big)^*\big(\psi(f\otimes\alpha)\otimes\psi(g\otimes\beta)\big)\\ &= \big(\psi(f\otimes\alpha)\otimes\psi(g\otimes\beta)\big)\circ\tau_{23}\circ(\Delta_C\otimes\Delta_D). \qquad (2)\end{aligned}$$

Now, for $f, g \in C^*, \alpha, \beta \in D^*, c \in C, d \in D$, (1) implies that

$$\begin{aligned}\big[\psi\big((f\otimes\alpha)(g\otimes\beta)\big)\big](c\otimes d) &= \big[\psi\big(\Delta^*_C(f\otimes g)\otimes\Delta^*_D(\alpha\otimes\beta)\big)\big]\\ &= [\Delta^*_C(f\otimes g)](c)[\Delta^*_D(\alpha\otimes\beta)](d)\\ &= \Big(\sum_c \underbrace{f(c_{(1)})\otimes g(c_{(2)})}_{\in k\otimes k=k}\Big)\Big(\sum_d \underbrace{\alpha(d_{(1)})\otimes\beta(d_{(2)})}_{\in k\otimes k=k}\Big)\\ &= \Big(\sum_c f(c_{(1)})g(c_{(2)})\Big)\Big(\sum_d \alpha(d_{(1)})\beta(d_{(2)})\Big)\\ &= \sum_{c,d} f(c_{(1)})g(c_{(2)})\alpha(d_{(1)})\beta(d_{(2)}),\end{aligned}$$

and (2) implies that

$$\begin{aligned}\big[\psi(f\otimes\alpha)\psi(g\otimes\beta)\big](c\otimes d) &= \big(\psi(f\otimes\alpha)\otimes\psi(g\otimes\beta)\big)\circ\tau_{23}\circ(\Delta_C\otimes\Delta_D)(c\otimes d)\\ &= \big(\psi(f\otimes\alpha)\otimes\psi(g\otimes\beta)\big)\\ &\quad\circ\tau_{23}\Big(\Big(\sum_c c_{(1)}\otimes c_{(2)}\Big)\Big(\sum_d d_{(1)}\otimes d_{(2)}\Big)\Big)\\ &= \big(\psi(f\otimes\alpha)\otimes\psi(g\otimes\beta)\big)\\ &\quad\circ\tau_{23}\Big(\sum_{c,d} c_{(1)}\otimes c_{(2)}\otimes d_{(1)}\otimes d_{(2)}\Big)\\ &= \big[\psi(f\otimes\alpha)\otimes\psi(g\otimes\beta)\big]\Big(\sum_{c,d} c_{(1)}\otimes d_{(1)}\otimes c_{(2)}\otimes d_{(2)}\Big)\\ &= \sum_{c,d}\big(\psi(f\otimes\alpha)(c_{(1)}\otimes d_{(1)})\big)\otimes\big(\psi(g\otimes\beta)(c_{(2)}\otimes d_{(2)})\big)\\ &= \sum_{c,d}\underbrace{f(c_{(1)})\alpha(d_{(1)})\otimes g(c_{(2)})\beta(d_{(2)})}_{\in k\otimes k=k}\\ &= \sum_{c,d} f(c_{(1)})\alpha(d_{(1)})g(c_{(2)})\beta(d_{(2)}).\end{aligned}$$

Noting that the two expressions agree (since k is commutative), we see that

$$\psi\big((f\otimes\alpha)(g\otimes\beta)\big) = \psi(f\otimes\alpha)\psi(g\otimes\beta).$$

Hence ψ is a homomorphism of algebras.