Math 201 Winter 2016 Homework 3

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In the first three exercises you fill in some of the details of the more elementary proof of the half of Gabriel's theorem which says that if a quiver Q has finite representation type, then Q is Dynkin.

1. Suppose that Q is a subquiver of a quiver Q'. Show that there is a functor $F : \operatorname{Rep}_K Q \to \operatorname{Rep}_K Q'$ defined as follows. For a represention (V, ϕ) of Q, F sends V to the representation (W, ψ) where $W_i = V_i$ if $i \in Q_0$, $W_i = 0$ if $i \in Q'_0 \setminus Q_0$, and on arrows, $\psi_\alpha = \phi_\alpha$ if $a \in Q_1$, $\psi_\alpha = 0$ if $a \in Q'_1 \setminus Q_1$. In other words, we "extend V by 0". I leave it to you to define the action on F on morphisms.

Now using F, prove that if Q has infinite representation type, then so does Q'.

We showed in class that every graph which is not Dynkin or Euclidean contains a Euclidean graph as a subgraph. Thus this exercise implies that to show that a non-Dynkin quiver has a infinite representation type, it suffices to show that quivers with underlying graph Euclidean have infinite representation type.

2. Suppose that a connected quiver Q has underlying graph $\overline{Q}$ which has a cycle; we include the case of a loop or a multiple edge. In other words, we assume that $\overline{Q}$ contains a copy of one of the Euclidean graphs $\widetilde{A}_m$, for any $m \geq 0$. To prove that Q has infinite representation type, it suffices by exercise 1 to prove that a quiver whose underlying graph is $\widetilde{A}_m$ has infinite representation type.

So assume now that $Q_0 = \{1, \ldots, m\}$ and there are m arrows in Q with α_i an arrow (of some orientation) between i and $i+1$ for $1 \leq i \leq m-1$, and α_m an arrow (of some orientation) between m and 1. In this case it is not hard to prove directly that Q has infinite representation type, as follows. For any $\lambda \in K$, define a representation (V, ϕ) with $V_i = K$ for all i; ϕ_{α_i} the identity map for all $1 \leq i \leq m-1$; and $\phi_{\alpha_m} : K \to K$ multiplication by λ.

Prove that the representations defined above are all indecomposable, and are pairwise nonisomorphic for distinct choices of λ. Thus assuming K is infinite, this shows that Q has infinite representation type.

By this exercise, a quiver of finite representation type cannot contain an unoriented cycle (where the notion of cycle includes loops and multiple edges). A connected graph with no cycles is also called a *tree*.

3. Suppose that Q is any connected quiver whose underlying graph $\overline{Q}$ is a tree. Let Q' be a quiver with the same underlying graph as Q. Show that there is a sequence of vertices $i_1, i_2, \ldots, i_m$ such that for each $j \geq 1$, i_j is a sink in $s_{i_{j-1}} \ldots s_{i_1}(Q)$, and where $s_{i_m} s_{i_{m-1}} \ldots s_{i_1}(Q) = Q'$.

This shows that the composition of reflection functors $C^+_{i_m} \ldots C^+_{i_1}$ is a functor from $\operatorname{Rep} Q$ to $\operatorname{Rep} Q'$. Using this and the properties of reflection functors we proved in class, show that Q has infinite representation type if and only if Q' does.

Given this and the preceding exercises, it now suffices to pick a single convenient orientation for each Euclidean graph of type D and E, and show that that quiver has infinite representation type. This case-by-case analysis can be found in Corollary 2.7 on page 259 of the book by Assem, Simson, and Skowronski and the preceding pages.

4. Let Q be a connected quiver whose underlying graph is a tree. Show that there is a numbering of the vertex set Q_0 of Q with $\{1, 2, \ldots, n\}$ such that for every arrow $a \in Q_1$, $h(a) < t(a)$. In other words, all arrows point from a larger number to a smaller number. Show then that i is always a sink of $s_{i-1} \ldots s_1(Q)$, so that one can define the composition of reflection functors $C^+ = C^+_n \ldots C^+_1 : \operatorname{Rep} Q \to \operatorname{Rep} Q$. Now let Q be have underlying graph which is Dynkin, and fix an admissible numbering of Q. Recall that for each vertex i, if e_i is the trivial path in the path algebra then KQe_i is an indecomposable projective module. The corresponding representation (P, ϕ) has $P_j = e_j KQ e_i$ for all j, with ϕ_α being left multiplication by α for any arrow $\alpha : j \to k$.

Show that if $S(i)$ is the simple representation of the quiver $s_i s_{i+1} \ldots s_n(Q)$ supported at vertex i, then $P(i) \cong C^-_1 \ldots C^-_{i-1}(S(i))$ and the dimension vector of $P(i)$ is $s_1 \ldots s_{i-1}(\epsilon_i)$

where ϵ_i is the dimension vector of $S(i)$. Also, formulate and prove a similar result for indecomposable injective representations.

5. As practice in understanding the definition of reflection functors, pick some orientation for the Dynkin graph E_6, number the vertices admissibly, consider the simple representation $S(i)$ supported at a sink i, and calculate what happens when you apply the composition of reflection functors $C = C_6^+ \dots C_1^+$ repeatedly.

6. Consider the quiver Q with $Q_0 = \{1, 2, 3, 4, 5\}$ and with edges $a : 2 \to 1$, $b : 3 \to 1$, $c : 4 \to 1$, $d : 5 \to 1$. The underlying graph of Q is the Euclidean graph $\widetilde{D}_4$. By Gabriel's theorem, Q has infinite representation type.

Consider the dimension vector $\beta = (2, 1, 1, 1, 1)$ (so the 2 is in the central vertex.) (a). Show that Q has infinitely many non-isomorphic representations with this dimension vector.

(b). In fact β is the vector that we showed spans the radical of the symmetrized Ringel form $(\ , \ )$. Show that $s_i(\beta) = \beta$ for any i, so applying a reflection functor C_i^+ to any representation with this dimension vector (of any quiver with this underlying graph) just gives another representation of the same dimension vector. If we perform $C_5^+ C_4^+ \dots C_1^+$ to a representation V with dimension vector β, is the representation we get isomorphic to the original one we started with?

7. Consider the representation space of the quiver Q with one vertex and one loop, with dimension vector (2). As we have seen, this is the space of 2×2-matrices, where two matrices M, N represent isomorphic representations if and only if they are conjugate, i.e. $M = PNP^{-1}$ for some $P \in \mathrm{GL}_2(K)$. Assuming that $K = \mathbb{C}$, there is one representation up to isomorphism for each 2×2 Jordan canonical form.

(a). Consider the orbits of the natural $\mathrm{GL}_2(K)$-action (i. e. similarity classes of matrices). For each one, what is its dimension as an algebraic variety? Which other orbits are in its closure?

(b). Generalize part (a) to $n \times n$ matrices for all $n \geq 2$, i.e. how does the dimension of each orbit depend on the Jordan form, and which similarity classes are in the closures of other similarity classes?