Math 201 Winter 2016 Homework 2

January 29, 2016

1. Suppose that Q is any quiver. The *opposite quiver* of Q is the quiver Q^{op} with the same vertex set as Q, and one arrow $\alpha^* : j \to i$ for each arrow $\alpha : i \to j$ in Q. In other words, Q^{op} is formed from Q by switching the direction of all of the arrows.

Given a left module M over KQ, show that there is a corresponding representation of Q^{op} given by putting $V_i = \epsilon_i M$ and $\phi_{\alpha^*} : V_j \to V_i$ given by left multiplication by $\alpha \in KQ$. Show that an anologous proof as the one we used for right modules gives an equivalence of categories between the category KQ-Mod of left KQ-modules and the category $\mathrm{Rep}_K(Q^{op})$ of representations of Q^{op}.

2. Fix any quiver Q and a field K. Let $\mathcal{C} = \mathrm{rep}_K Q$ be the category of *finite-dimensional* representations of Q (note the lowercase r on rep) and let $\mathcal{D} = \mathrm{rep}_K Q^{op}$ be the category of finite-dimensional representations of Q^{op}.

(a). Show that there is a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ which does the following. Given a rep (V, ϕ) of Q, $F(V, \phi)$ is the rep (V^*, ϕ^*) of Q^{op} which has $(V^*)_i = (V_i)^* = \mathrm{Hom}_K(V_i, K)$ and for $\alpha : i \to j$ in Q, has $\phi^*_{\alpha^*} = (\phi_\alpha)^* : V_j^* \to V_i^*$, that is, $(\phi_\alpha)^*(\psi) = \psi \circ \phi_\alpha$ for $\psi : V_j \to K$. The action of F on morphisms is the obvious one; I leave it to you to define it.

Prove that F is a duality (that is, there is a contravariant functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to identity functors on $\mathcal{D}$ and $\mathcal{C}$ respecitvely.)

(b). We proved in class that there is an equivalence of categories $H : \mathrm{rep}_K(Q) \to$ fd-KQ where fd-KQ means the category of finite-dimensional right KQ-modules. We also proved there is a duality D : fd-$KQ \to KQ$-fd, where KQ-fd is the category of finite-dimensional left KQ-modules. By problem 1, there is an equivalence of categories J :

KQ-fd $\to \mathrm{rep}_K(Q^{op})$. Show that $J \circ D \circ H$ is naturally isomorphic to the duality F described in part (a).

3. Consider the acyclic quiver Q given by

$$\begin{tikzcd}[row sep=small]
\overset{1}{\bullet} \arrow[dr] & & & & \\
& \overset{2}{\bullet} \arrow[r] & \overset{4}{\bullet} \arrow[r] & \overset{5}{\bullet} \arrow[r] & \overset{6}{\bullet} \\
\underset{}{\overset{3}{\bullet}} \arrow[ur] & & & &
\end{tikzcd}$$

Find explicitly all simple, indecomposable projective, and indecomposable injective representations of this quiver.

4. Verify the details of how Hom sets get bimodule structures, as follows:

(a). If M is a (B, A)-bimodule and N is an (C, A)-bimodule, show that $\mathrm{Hom}_A(M, N)$ is a (C, B)-bimodule where $[c\phi](m) = c(\phi(m))$ and $[\phi b](m) = \phi(bm)$.

(b). If M is an (A, B)-bimodule and N is an (A, C)-bimodule, show that $\mathrm{Hom}_A(M, N)$ is a (B, C)-bimodule where $[b\phi](m) = \phi(mb)$ and $[\phi c](m) = \phi(m)c$.

5. Let Q be the Kronecker quiver

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & \overset{2}{\bullet}
\end{tikzcd}$$

Let $A = KQ$. Let (V, ϕ) be the (infinite-dimensional) representation of Q given by $V_1 = V_2 = K[t]$ (the polynomial ring in one variable) with ϕ_α the identity map and ϕ_β multiplication by t. Let M be the right KQ-module corresponding to the rep (V, ϕ).

Show that M is indecomposable, but that $\mathrm{End}\, M$ is not local. Thus endomorphism rings of indecomposable modules which are not of finite length need not be local, even over a finite-dimensional algebra.

6. Let Q be an acylic quiver with vertices $1, \dots n$.

(a). Let P be a finite-dimensional projective right module over KQ, and let (V, ϕ) be the corresponding representation. Show that for all arrows α in Q, the map ϕ_α in the representation is an injective vector space map. (Hint: reduce to the indecomposable case).

(b). Similarly, if E is a finite-dimensional injective right module over KQ, show that the maps in the corresponding rep (V, ϕ) are all surjective.