Math 201 Winter 2016 Homework 1

January 15, 2016

1. Fill in the missing details of the proofs given in class this past week that there is a 1-to-1 correspondence between right modules over KQ and representations of Q over K; and a 1-to-1 correspondence between homomorphisms $M \to N$ of right KQ-modules and morphisms of the corresponding quiver representations (V, ϕ) and (W, ψ).

2. This exercise assumes some familiarity with the language of categories and functors. We will review this language later in class. The point of this exercise is to show a formal context that subsumes both representations of quivers and representations of groups.

(a). Given a quiver Q and a field K, define a category C whose objects are the vertices in Q, and which has one morphism $i \to j$ for each path in Q with source i and target j. Composition of morphisms is given by composition of paths. (Note that the identity morphisms of each object are given by the trivial paths).

Prove that the represention of Q over K are in 1-to-1 correspondence with functors F from C to the category of K-vector spaces.

(b). Given a group G, define a category D with one object X and one arrow $X \to X$ for each group element $g \in G$. Composition of paths is given by multiplication in G.

Prove that representations of G over K are in 1-to-1 correspondence with functors F from D to the category of K-vector spaces.

3. (a). Let Q be a quiver without cycles. Prove that there is one simple rep of Q over K for each vertex $i \in Q_0$, given by $V_i = K$ and $V_j = 0$ for $j \neq i$, (note that all arrows automatically have the 0 map associated to them), and that these are all of the simple representations up to isomorphism.

(b). Let Q be a quiver having at least one cycle. Prove that there are infinitely many pairwise non-isomorphic simple representations of Q.

4. let Q be the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \dots \qquad \xrightarrow{\alpha_{n-1}} n$$

For any $0 \leq i \leq j \leq n$, we have a representation of Q of the form

$$0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \underset{i}{K} \xrightarrow{1} K \xrightarrow{1} \dots \xrightarrow{1} \underset{j}{K} \longrightarrow 0 \longrightarrow \dots \longrightarrow 0.$$

Show that these are pairwise non-isomorphic indecomposable representations. Then generalizing our example in class (where $n = 2$), show that these are all of the indecomposable representations of Q up to isomorphism.

We will see later that the classification of indecomposable reps over this quiver also follows from a more general abstract theorem which characterizes those quivers which have finitely many indecomposable reps up to isomorphism.

5. Let Q be the Kronecker quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2\,.$$

Assume the base field K is algebraically closed. Find all simple representations of Q. Then find all indecomposable representations (V, ϕ) up to isomorphism for which both $\dim_K V_1 \leq 2$ and $\dim_K V_2 \leq 2$. (Some of this was done in class, but not all details were given).

6. Same questions as #5, but for the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2\,.$$

(By problem 3, there are infinitely many simples up to isomorphism.)

7. (a). Let Q be a quiver without cycles. Show that you can choose a labeling $\{1, 2, \ldots, n\}$ of the vertex set of Q such that if there exists a path from i to j, then $i \leq j$.

(b). Let Q be a quiver with the property that for any vertices i, j, there exists at most one path (including trivial paths) from i to j. In paticular, Q has no cycles. Choose a labeling as in part (a).

Prove that KQ is isomorphic to a subalgebra of the upper triangular matrix algebra

$$T = \{(a_{ij}) \in M_n(K) | a_{ij} = 0 \text{ if } i > j\}.$$