Math 201a Spring 2017 Homework 4

Due June 9, 2017

I will ask for volunteers to present (brief) sketches of solutions to some of the problems in class, Friday June 9, 2017.

1. Let $R = k[x, y]$ for a field k. Let k be the simple module $R/(x, y)$.

(a). Calculate $\mathrm{Ext}^1_R(k, k)$. Using the correspondence between Ext and extensions, classify up to isomorphism all length 2 R-modules M with composition factors k and k.

(b). Classify up to isomorphism all length 3 R-modules M with composition factors k, k, and k.

2. Let R and S be rings and B an (R, S)-bimodule. We can form the upper triangular matrix ring $A = \begin{bmatrix} R & B \\ 0 & S \end{bmatrix}$ where the ring product is

$$\begin{bmatrix} r_1 & b_1 \\ 0 & s_1 \end{bmatrix} \begin{bmatrix} r_2 & b_2 \\ 0 & s_2 \end{bmatrix} = \begin{bmatrix} r_1 r_2 & r_1 b_2 + b_1 s_2 \\ 0 & s_1 s_2 \end{bmatrix}.$$

(a). Show that right A-modules are in bijection with triples (M, N, f) where $M \in \text{Mod-}R$, $N \in \text{Mod-}S$, and $f \in \mathrm{Hom}_S(M \otimes_R B, N)$. (Hint: If P is a right A-module, let $M = Pe_1$ and $N = Pe_2$, where $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Conversely, given the data (M, N, f) show how to reconstruct a right A-module.)

(b). Show that if P is a projective right A-module and (M, N, f) is the corresponding triple, then N is a projective right S-module.

(c). If $A = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{bmatrix}$, show that A is not right hereditary.

(d). Let A be the same ring as in part (c). Show that A is left hereditary.

3. (a). If M is a left R-module such that pd(M) $= n < \infty$, show that there is a free module F such that $\mathrm{Ext}^n_R(M, F) \neq 0$.

(b). Suppose that R is a ring which is noetherian and self-injective (that is, R is injective as a left R-module). If $n = \mathrm{l.gl.dim}(R)$, show that either $n = 0$ or $n = \infty$. Give examples of both kinds. (Hint: recall that over a noetherian ring an arbitrary direct sum of injective modules remains injective).

4. Let $R = \mathbb{Z}_n$ be the ring of integers mod n. Find $\mathrm{gl.dim}\, R$ (as a funtion of n). (Problem 3 could be useful).

5. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of left R-modules.

(a). Prove that $\mathrm{pd}(N) \leq 1 + \max(\mathrm{pd}(L), \mathrm{pd}(M))$.

(b). If M is projective, prove that either L and N are also projective or else $\mathrm{pd}(N) = 1 + \mathrm{pd}(L)$.

6. (a). Given a family of left R-modules $\{M_i\}_{i \in I}$, prove that $\mathrm{pd}(\bigoplus_i M_i) = \sup_i \mathrm{pd}(M_i)$.

(b). Show that if $\mathrm{l.gldim}(R) = \infty$, then there exists a left R-module M with $\mathrm{pd}(M) = \infty$.

7. Let $R = k\langle x, y \rangle$ be a free associative algebra in two variables over the field k. Show that R is (left) hereditary by proving that every left ideal of R is free.