Math 201a Spring 2017 Homework 3

Due May 12, 2017

I will ask for volunteers to present (brief) sketches of solutions to some of the problems in class, Friday May 12, 2017.

1. Let (F, G) be an adjoint pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, where $\mathcal{C}$ and $\mathcal{D}$ are any categories. let $\rho_{M,N} : \mathrm{Hom}_{\mathcal{D}}(FM, N) \to \mathrm{Hom}_{\mathcal{C}}(M, GN)$ be the isomorphisms giving the adjointness.

(a). For each $M \in \mathcal{C}$, $\rho_{FM,FM}(1_{FM})$ is a morphism $\eta_M : M \to GFM$. Show that the maps η_M determine a natural transformation of functors $\eta : 1 \to GF$ (here 1 indicates the identity functor on $\mathcal{C}$.) η is called the *unit* of the adjunction.

(b). Show how to determine in a dual way to part (a) a natural transformation of functors $\epsilon : FG \to 1$ (here 1 indicates the identity functor on $\mathcal{D}$). ϵ is called the *counit* of the adjunction.

(c). Suppose that $f \in \mathrm{Hom}_{\mathcal{D}}(FM, N)$. Show that $\rho_{FM,N}(f) = G(f) \circ \eta_M$. Similarly, if $g \in \mathrm{Hom}_{\mathcal{C}}(M, GN)$, show that $\rho^{-1}_{FM,N}(g) = \epsilon_N \circ F(g)$. Conclude that $\epsilon_{FM} \circ F(\eta_M) = 1_{FM}$ for all $M \in \mathcal{C}$ and that $G(\epsilon_N) \circ \eta_{GN} = 1_{GN}$ for all $N \in \mathcal{D}$.

2. Suppose one is given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, together with morphisms of functors $\eta : 1 \to GF$ and $\epsilon : FG \to 1$ which satisfy $\epsilon_{FM} \circ F(\eta_M) = 1_{FM}$ for all $M \in \mathcal{C}$ and $G(\epsilon_N) \circ \eta_{GN} = 1_{GN}$ for all $N \in \mathcal{D}$. Show that (F, G) are an adjoint pair.

3. (a). Let $R = k[x]$ where k is a field. Let k indicate the R-module $k[x]/(x)$. Calculate $\mathrm{Ext}^1_R(k, R)$ by using a projective resolution of the first coordinate.

(b). With the same notation as in part (a), calculate $\mathrm{Ext}^1_R(k, R)$ by using an injective resolution of the second coordinate.

(c). Now let $R = k[x, y]$ and let $k = k[x, y]/(x, y)$. What is $\mathrm{Ext}^1_R(k, R)$ in this case? Use whichever resolution is most convenient.

4. Recall that if F is not left exact, then the right derived functor R^0F will not be the same as (or more correctly, naturally isomorphic to) F. Similarly, if F is not right exact, then L_0F is different from F.

Let R be a commutative ring with a nonzero ideal I. Define a functor $F : R\text{-Mod} \to R\text{-Mod}$ by $F(M) = IM$, where if $f : M \to N$ is a morphism, then $F(f) : IM \to IN$ is just the restriction of f.

(a). Show that F is additive but not right exact in general.

(b). Give an explicit description of the functor L_0F, expressing it in terms of some more familiar functor.

5. Let $\phi : R \to S$ be a ring homomorphism. Consider S as a left R-module via ϕ, and assume that S is flat as a right R-module. Prove that

$$\mathrm{Tor}^R_i(M, N) \cong \mathrm{Tor}^S_i(M, S \otimes_R N)$$

For all $M \in \text{Mod-}S$, $N \in R\text{-Mod}$.

6. Let R be a commutative domain with field of fractions Q. Let M be an R-module whose annihilator is nonzero, i.e. there exists $0 \neq r \in R$ such that $rm = 0$ for all $m \in M$.

(a). Prove that $\mathrm{Ext}^i_R(M, Q) = 0$ for all $i \geq 0$.

(b). Prove that $\mathrm{Tor}^R_i(Q, M) = 0$ for all $i \geq 0$.

(c). Is $\mathrm{Ext}^i_R(Q, M) = 0$ for all $i \geq 0$?

7. Suppose that that $0 \to K \to M \to P \to 0$ is a short exact sequence of left R-modules.

(a). If M and P are flat, prove that K is flat.

(b). Is it true more generally that if any two of the modules in the sequence are flat, then the third one is?

8. Prove that the functor $T = \mathrm{Tor}^{\mathbb{Z}}_1(G, -)$ is left exact for every Abelian group G, and compute its right derived functors L_nT.