Math 201a Spring 2017 Homework 2

Due April 28, 2017

I will ask for volunteers to present (brief) sketches of solutions to some of the problems in class, Friday April 28.

In problems 1 and 2 below, you may want to start by trying to find the product or coproduct of finitely many objects in the category, before tackling the case of an arbitrary indexed family of objects.

1. The category of *pointed sets* $\mathcal{C}$ is the category whose objects are ordered pairs (X, p) where X is a set and $p \in X$, and where a morphism $(X, p) \to (Y, q)$ is a function $f : X \to Y$ such that $f(p) = q$.

(a). Does an arbitrary (possibly infinite) indexed family of pointed sets $\{(X_\alpha, p_\alpha)\}_{\alpha \in I}$ have a product in $\mathcal{C}$? If so, describe it.

(b). Does an arbitrary (possibly infinite) indexed family of pointed sets $\{(X_\alpha, p_\alpha)\}_{\alpha \in I}$ have a coproduct in $\mathcal{C}$? If so, describe it.

2. let k be a field. Recall that a k-algebra is a ring R together with a k-vector space structure on R such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in k$, $a, b \in R$.

Consider the category $\mathcal{C}$ of commutative k-algebras, with morphisms the homomorphisms of algebras (i.e. ring homomorphisms that are also linear transformations of k-vector spaces).

(a). Does an arbitrary (possibly infinite) indexed family of k-algebras $\{R_\alpha\}_{\alpha \in I}$ have a product in $\mathcal{C}$? If so, describe it.

(b). Does an arbitrary (possibly infinite) indexed family of k-algebras $\{R_\alpha\}_{\alpha \in I}$ have a coproduct in $\mathcal{C}$? If so, describe it.

3. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Suppose that products and coproducts of any indexed families exist in $\mathcal{C}$ and $\mathcal{D}$.

Given a family $\{M_\alpha\}_{\alpha \in I}$ of objects in $\mathcal{C}$, consider the coproduct $\coprod_{\alpha \in I} M_\alpha$ together with its injection morphisms $i_\alpha : M_\alpha \to \coprod_{\alpha \in I} M_\alpha$. Applying the functor F we have morphisms $F(i_\alpha) : F(M_\alpha) \to F(\coprod_{\alpha \in I} M_\alpha)$ for all α, and thus by the universal property of the coproduct, a uniquely determined morphism $f : \coprod_{\alpha \in I} F(M_\alpha) \to F(\coprod_{\alpha \in I} M_\alpha)$ such that $f \circ j_\alpha = F(i_\alpha)$, where $j_\alpha : F(M_\alpha) \to \coprod_{\alpha \in I} F(M_\alpha)$ is the injection morphism associated with this coproduct in $\mathcal{D}$. We say that F *commutes with coproducts* if f is an isomorphism for all indexed families.

Similarly, since $\mathcal{C}$ and $\mathcal{D}$ have products for any indexed family $\{N_\alpha\}$ of objects in $\mathcal{D}$ there is a canonically defined morphism $g : G(\prod_\alpha N_\alpha) \to \prod_\alpha G(N_\alpha)$ and we say that G *commutes with products* if g is an isomorphism for all indexed families.

Now suppose that (F, G) are an adjoint pair. Show that F commutes with coproducts and G commutes with products. Use this to give another proof that tensor products commute with direct sums.

(Hint: consider the case of products, for example. Show for any object $L \in \mathcal{C}$ that there is an isomorphism of sets

$$\mathrm{Hom}_{\mathcal{C}}(L, G(\prod N_\alpha)) \to \mathrm{Hom}_{\mathcal{C}}(L, \prod G(N_\alpha))$$

and that this is the same map as $(-)\circ g$. Then conclude that g must itself be an isomorphism.)

4. Let R be a ring. The (left) Grothendieck group of R, $K_0(R)$, is defined as follows. Consider the set of finitely generated projective left R-modules, and put an equivalence relation on this set where two modules are equivalent if they are isomorphic. Let $[P]$ be the equivalence class of P under this relation. Now Let F be the free Abelian group generated by the distinct equivalence classes $[P]$. Then define $K_0(R) = F/I$, where I is the subgroup generated by all elements of the form $[P] + [Q] - [P \oplus Q]$ as P and Q range over finitely generated projectives.

(a). Show that if R is a ring such that all finitely generated projectives are free, then $K_0(R) \cong \mathbb{Z}$. Conclude that $K_0(R) = \mathbb{Z}$ for any principal ideal domain.

(b). A projective module P is called *stably free* if there exists a finitely generated free module F such that $P \oplus F$ is free. Show that if every finitely generated projective R-module is stably free, then $K_0(R) \cong \mathbb{Z}$ again.

(c*) (optional, for those familiar with the theory of Dedekind domains). Let $R = \mathbb{Z}[\sqrt{-5}]$. Find $K_0(R)$. (Hint: In a Dedekind domain, every ideal is projective. The Grothendieck group is closely related to the class group $\mathrm{Cl}(R)$.)

5. Let R be a commutative domain. Let F be the field of fractions of R, considered as an R-module by multiplication. Show that F is an injective R-module.

6. Let k be a field.

(a). A ring R is called *self-injective* if R is injective as a left R-module. Prove that the ring $k[x]/(x^n)$ is self-injective, for any $n \geq 1$.

(b). Let R be a k-algebra which is finite-dimensional over k. Notice that $E = \mathrm{Hom}_k(R, k)$ obtains a left R-structure since R is an (k, R)-bimodule. The algebra R is called *Frobenius* if $E \cong R$ as left R-modules. Show that if R is Frobenius, then it is self-injective. (Hint: R is projective as a module and the functor $\mathrm{Hom}_k(-, k)$ is contravariant and so reverses the directions of arrows.)

(c). Show that $R = k[x, y]/(x^2, y^2)$ is Frobenius and hence self-injective.

7. Given a module M, recall that the *injective dimension* of M is the smallest $n \geq 0$ such that there exists an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ where the E_i are injective modules. Dually, the *projective dimension* of M is the smallest $n \geq 0$ such that there exists an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ where the P_i are projective modules.

(a). Let R be a PID which is not a field. Show that every module M over R has injective dimension 0 or 1. Which finitely generated modules M have injective dimension 0 and which have injective dimension 1?

(b). Again let R be a PID which is not a field. Show that every module M over R has projective dimension 0 or 1. Which finitely generated modules M have projective dimension 0 and which have projective dimension 1?