Math 201a Spring 2017 Homework 1

Due April 14, 2017

1 Instructions

The first section of problems here are routine problems are for you to work through to check your understanding. Many of them are checking details of various statements made in class (which you should do anyway as you go over your notes between classes). The second section of problems are less routine. I will ask for volunteers to present (brief) sketches of solutions to some of the problems from the second section in class, probably Friday April 14.

2 Routine Exercises

1. Given a category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{op}$ is a category with the same objects as $\mathcal{C}$, but with Hom sets defined by $\mathrm{Hom}_{\mathcal{C}^{op}}(M, N) = \mathrm{Hom}_{\mathcal{C}}(N, M)$. Composition is done in reverse from composition in $\mathcal{C}$, so that if $f \in \mathrm{Hom}_{\mathcal{C}^{op}}(M, N)$, $g \in \mathrm{Hom}_{\mathcal{C}^{op}}(P, M)$, then $f \circ_{\mathcal{C}^{op}} g = g \circ_{\mathcal{C}} f$. The identity morphism in $\mathrm{Hom}_{\mathcal{C}^{op}}(M, M)$ is the identity morphism in $\mathrm{Hom}_{\mathcal{C}}(M, M)$.

(a). Check that $\mathcal{C}^{op}$ is a category.

(b). Let R be a ring and let $\mathcal{C} = \mathcal{C}_R = \{X\}$ be the one object category with $\mathrm{Hom}_{\mathcal{C}}(X, X) = R$, where $f \circ g = fg$ for $f, g \in R$. Show that $\mathcal{C}^{op}$ is the same as $\mathcal{C}_{R^{op}}$.

(c). Show that a contravariant functor $\mathcal{C} \to \mathcal{D}$ is the same concept as a covariant functor $\mathcal{C}^{op} \to \mathcal{D}$.

2. Let $\mathcal{C} = \mathcal{C}_R = \{X\}$ be the one-object category associated to a ring R, as in exercise 1(b). Show that there is a one-to-one correspondence between left R-modules and additive functors $F : \mathcal{C} \to \mathrm{Ab}$. (Recall that if $\mathcal{C}$ and $\mathcal{D}$ are preadditive categories, a functor $F : \mathcal{C} \to \mathcal{D}$

is called additive if $F(-) : \mathrm{Hom}_{\mathcal{C}}(M, N) \to \mathrm{Hom}_{\mathcal{D}}(FM, FN)$ is a homomorphism of Abelian groups, for all M and N).

3. Check carefully the claim made in class that given a ring R, there is a functor $H_n : \mathrm{Ch}(R\text{-Mod}) \to R\text{-Mod}$ given by taking the nth Homology group of a chain complex.

4. Check the details of the fact that bimodules give Hom groups extra structure. Namely, given an (R, S)-bimodule M and a (R, T)-bimodule N, then $\mathrm{Hom}_{R\text{-Mod}}(M, N)$ is an (S, T)-bimodule. Check also that given an (S, R)-bimodule M and a (T, R)-bimodule N then $\mathrm{Hom}_{\text{Mod-}R}(M, N)$ is a (T, S)-bimodule.

5. Check the details of the adjoint isomorphism between Hom and tensor product as presented in class.

3 Less routine exercises

6. Let $M, N, P \in R\text{-Mod}$ and let $M \xrightarrow{f} N \xrightarrow{g} P$ be a sequence of morphisms in R-Mod with $g \circ f = 0$. Show that the following are equivalent:

(a) $0 \to M \xrightarrow{f} N \xrightarrow{g} P$ is exact;

(b) $0 \to \mathrm{Hom}_R(Q, M) \xrightarrow{f\circ-} \mathrm{Hom}_R(Q, N) \xrightarrow{g\circ-} \mathrm{Hom}_R(Q, P)$ is exact for all modules $Q \in R\text{-Mod}$.

Similarly, show that the following are equivalent:

(a) $M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is exact;

(b) $0 \to \mathrm{Hom}_R(P, Q) \xrightarrow{-\circ g} \mathrm{Hom}_R(N, Q) \xrightarrow{-\circ f} \mathrm{Hom}_R(M, Q)$ is exact for all modules $Q \in R\text{-Mod}$.

Conclude in particular that the functors $\mathrm{Hom}_{R\text{-Mod}}(M, -)$ and $\mathrm{Hom}_{R\text{-Mod}}(-, M)$ are (covariant and contravariant, respectively) left exact functors $R\text{-Mod} \to \mathrm{Ab}$.

7. Show that the categories $\mathcal{C} = \mathrm{Set}$, $\mathcal{C} = \mathrm{Ring}$, and $\mathcal{C} = \mathrm{Group}$ are not pre-additive categories. In other words, for each $\mathcal{C}$, there is no way to assign an Abelian group structure

to each set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that composition in the category is bilinear. (The argument may be different for each category).

8. (a). Let $G : R\text{-Mod} \to \text{Set}$ be the functor which sends a module to its underlying set, sometimes called a *forgetful functor* since we are remembering only part of the structure. Show that there is a functor $F : \text{Set} \to R\text{-Mod}$ which sends a set X to the free left R-module with basis X. Show that (F, G) are an adjoint pair of functors.

(b). Now let $G : R\text{-Mod} \to \text{Ab}$ be the functor which sends a module to its underlying Abelian group (so this is a slightly less forgetful functor). Does G have a left adjoint F? If so, describe it and prove it is an adjoint.

9. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(M, N)$ in a category $\mathcal{C}$ is called an *isomorphism* if there is $g \in \mathrm{Hom}_{\mathcal{C}}(N, M)$ such that $f \circ g = 1_N$ and $g \circ f = 1_M$. In this case we say that objects M and N are isomorphic and write $M \cong N$. Thus the isomorphisms in the categories Set, Ring, R-Mod, and Group are just the usual isomorphisms.

(a). Let $\mathcal{C}$ be the category with one object X and $\mathrm{Hom}_{\mathcal{C}}(X, X) = R$ for a ring R, where composition in $\mathcal{C}$ is multiplication in R (as in Exercise 1(b)). Which morphisms are isomorphisms in this category?

(b). Define a category $\mathcal{C}$ as follows. The objects of $\mathcal{C}$ are all of the rings. The set of morphisms $\mathrm{Hom}_{\mathcal{C}}(R, S)$ is defined to be the set of all isomorphism classes of (R, S)-bimodules. (That is, if two (R, S)-bimodules are isomorphic as bimodules, we consider them to be the same morphism). Given ${}_RM_S \in \mathrm{Hom}_{\mathcal{C}}(R, S)$ and ${}_SM_T \in \mathrm{Hom}_{\mathcal{C}}(S, T)$, then we define the composition of these in $\mathcal{C}$ to be $M \otimes_S N$, which has the structure of an (R, T)-bimodule and so is in $\mathrm{Hom}_{\mathcal{C}}(R, T)$. It is easy to see that this is a category.

What does it mean for two rings R and S to be isomorphic in this category $\mathcal{C}$? Write this as an explicit condition involving bimodules.

(c). Continuing with the category $\mathcal{C}$ in part (b), let k be a field. Show that the rings k and $M_2(k)$ (2×2 matrices over k) are isomorphic objects in the category $\mathcal{C}$.