

Math 200c Spring 2012 Homework 4

Due 6/8/2012 by 5pm in homework box (now in basement)

I will consider four out of the five possible scores in this class when assigning grades (4 homeworks and the qual). If you are taking the qual and are happy with your first three homework scores, you could consider handing in this homework optional. Students not taking the qual should hand in all four homeworks.

Assigned problems (all to be turned in).

1. Recall the following definitions.

A domain A is a valuation ring if for all $x \in K$, the field of fractions of A , either $x \in A$ or $x^{-1} \in A$.

An abelian group Γ is called totally ordered if the elements of Γ are a poset which is totally ordered (for all $\alpha, \beta \in \Gamma$, either $\alpha \leq \beta$ or $\beta \leq \alpha$) and compatible with the addition ($\alpha \leq \beta \implies \alpha + \gamma \leq \beta + \gamma$ for all $\alpha, \beta, \gamma \in \Gamma$). Note that we take Γ to be written additively and write 0 for its identity element.

A valuation of a field K is a surjective map $v : K^\times \rightarrow \Gamma$, where Γ is a totally ordered abelian group called the value group, such that

- (i) $v(ab) = v(a) + v(b)$ for all $a, b \in K^\times$; and
- (ii) $v(a + b) \geq \min(v(a), v(b))$ for all $a, b \in K^\times$.

(a). Show that if $v : K^\times \rightarrow \Gamma$ is a valuation, then $A = \{x \in K | v(x) \geq 0\}$ is a valuation ring.

(b). If A is a valuation ring, define $\Gamma = K^\times / U$, where $K^\times$ is the multiplicative group of the field and U is the group of units of A . Show that Γ is a totally ordered Abelian group, if we define $xU \leq yU$ if $yx^{-1} \in A$. (Note that in the definitions above we wrote Γ additively, but this Γ is more naturally written multiplicatively.) Show that the map $v : K^\times \rightarrow \Gamma$ given by $v(x) = xU$ is a valuation.

(c). Show that if A is a valuation ring and we define the valuation v as in part (b), then $A = \{x \in K | v(x) \geq 0\}$. Conversely, if $v : K^\times \rightarrow \Gamma$ is a valuation, and we use

$A = \{x \in K \mid v(x) \geq 0\}$ to define a valuation $w : K^\times \rightarrow \Gamma'$ as in part (b), then there is an order preserving isomorphism of the value groups $\phi : \Gamma \rightarrow \Gamma'$ such that $\phi \circ v = w$. Thus valuation rings and valuations are essentially equivalent concepts.

2. Let A be a valuation ring which is not a field. Then A is noetherian if and only if it is a discrete valuation ring. (Hint: Show if A is noetherian that the corresponding value group is isomorphic to $\mathbb{Z}$).

3. Let A be a finitely generated k -algebra for some field k . Prove that the following are equivalent:

- (i) $\dim_k A < \infty$.
- (ii) A is an artinian ring.

4. Let A be a Dedekind domain, and S a multiplicative system in A .

(a). Show that $S^{-1}A$ is either again a Dedekind domain, or else is the entire field of fractions of A . (Hint: this is mostly quoting theorems.)

(b). Show that extension of ideals induces a surjective homomorphism from the ideal class group of A to the ideal class group of $S^{-1}A$.

5. Let A be a Dedekind domain. Prove that every ideal I of A is generated by at most 2 elements.

(Hint: For any nonzero $a \in A$, consider the ring $A/(a)$; one needs to show that every ideal in this ring is principal. We have $(a) = P_1^{i_1} \dots P_n^{i_n}$ for some maximal ideals P_j and $i_j \geq 1$. Show that as rings, $A/(a) \cong \prod_j A/P_j^{i_j}$. Prove that each $A/P_j^{i_j}$ is a ring in which all ideals are principal, and conclude that every ideal of $A/(a)$ is principal.)