MATH 200B WINTER 2022 MIDTERM SOLUTIONS

1 (15 pts). Let $A \in M_3(F)$ where F is an algebraically closed field. Classify up to similarity the matrices A which satisfy $A^3 = A$, by describing their possible Jordan forms. There may be cases depending on the characteristic of F. Make sure you justify your answer.

Solution.

Because $A^3 = A$, the matrix A satifies the polynomial $x^3 - x = x(x-1)(x+1)$. Thus the minimal polynomial of A divides this.

Case 1: $\operatorname{char} F \neq 2$. When the characteristic is not 2, the numbers $-1, 0, 1$ are distinct and so the primes x, $x-1$, $x+1$ are pairwise non-associate. The invariant factors of A all divide the minimal polynomial, which is the largest invariant factor. So every invariant factor is also a product of distinct primes in $F[x]$. The elementary divisors are found by splitting each invariant factor as a product of powers of distinct primes. Thus every elementary divisor is linear and is equal to x, $x-1$, or $x+1$. The Jordan form of A is then of the form $J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ where each $\lambda_i \in \{-1, 0, 1\}$. Conversely, it is obvious that every such Jordan form J satisfies $J^3 = J$, and thus every matrix A similar to J satisfies $A^3 = A$. So the set of possible matrices A is the set of all conjugates of these J.

This shows every such A is in the union of certain similarity classes. To be more exact, one can say how many distinct similarity classes of matrices there are: since Jordan forms are similar if and only if they are the same after rearranging blocks, there are 3 classes in which all λ_i are equal; 6 classes in which two of the λ_i are equal; and 1 class in which all λ_i are distinct, so there are 10 classes total.

Case 2: $\operatorname{char} F = 2$. In this case $x^3 - x = x(x-1)^2$ and so the minimal polynomial is not necessarily a product of distinct primes. Since each invariant factor divides this, splitting the invariant factors into elementary divisors we find that each elementary divisor is either x, $x-1$, or $(x-1)^2$. Thus every Jordan form of A is either of the form $J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ or else of the form $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$, where each $\lambda_i \in \{0, 1\}$. Conversely it is easy to calculate that every such J satisfies $J^3 = J$, so the set of possible A is the set of all conjugates of these J.

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Counting the number of similarity classes, one has 2 diagonal matrices in which the λ_i are all equal and 2 in which two of them are equal. There are 2 that contain the Jordan block. Thus there are 6 classes total.

2 (15 pts). Let R be a PID and M a nonzero finitely generated torsion left R-module. Recall that M is *uniserial* if there is a finite chain of submodules

$$M_0 = \{0\} \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that the M_i are all of the R-submodules of M.

(a) (10 pts) Show that M has only one elementary divisor if and only if M is uniserial.

Solution.

(The hypothesis that M is finitely generated was inadvertently left out of the original version. The elementary divisors and invariant factors are not even defined for an infinitely generated torsion module.)

Let $p_1^{e_1}, \ldots, p_m^{e_m}$ be the elementary divisors of M, so that $M \cong R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_m^{e_m})$. Suppose that $m \geq 2$, that is that there is more than one elementary divisor. In particular, we can decompose M as a nontrivial direct sum $M = N \oplus P$ with $N \neq 0$ and $P \neq 0$ (take $N = R/(p_1^{e_1})$ and $P = R/(p_2^{e_2}) \oplus \cdots \oplus R/(p_m^{e_m})$). But no such nontrivial direct sum can be a uniserial module, because thinking of M as an internal direct sum of N and P, then N and P are submodules of M such that $N \cap P = 0$, and thus neither $N \subseteq P$ nor $P \subseteq N$ holds. So it is impossible to write all of the submodules of M in a single chain.

Conversely, suppose that M has one elementary divisor p^e for a prime p, so $M = R/(p^e)$. Since this is a factor module of R, its submodules are are in bijective correspondence with submodules I of R (that is, ideals) such that $(p^e) \subseteq I \subseteq R$. Since R is a PID, $I = (a)$ is principal and $(p^e) \subseteq (a)$ is equivalent to $a|p^e$. Since p is prime, by unique factorization the only divisors of p^e (up to associates) are the elements p^i with $0 \leq i \leq e$. So $I = (p^i)$ for some such i. Then the possible I form a single chain $(p^e) \subseteq (p^{e-1}) \subseteq \cdots \subseteq (p) \subseteq R$. Then by submodule correspondence, the submodules of M also form a single chain $(0) \subseteq (p^{e-1})/(p^e) \subseteq \cdots \subseteq (p)/(p^e) \subseteq R/(p^e)$ and M is uniserial.

(b) (5 pts) Show that M has only one invariant factor if and only if M is cyclic.

We have $M \cong R/(f_1) \oplus R/(f_2) \oplus \cdots \oplus R/(f_k)$ where each f_i is nonzero, nonunit, and $f_i | f_{i+1}$ for $0 \leq i \leq k-1$. Then $f_1, f_2, \ldots, f_k$ is the sequence of invariant factors of M.

Suppose that M has a single invariant factor. Then $k = 1$ and $M \cong R/(f_1)$. The module $R/(f_1)$ is cyclic, generated by $1 + (f_1)$.

Conversely, suppose that M is cyclic. As we saw in class, a cyclic module is isomorphic to R/I for some ideal I. (you can quote this without proof. But here is why: suppose that M is generated by m. Define a homomorphism $\phi : R \to M$ by $\phi(r) = rm$. Then ϕ is surjective.

The kernel of ϕ is a submodule of R, that is, an ideal I. By the 1st isomorphism theorem, $M \cong R/I$.) Since R is a PID, $I = (g)$ for some g. Since $M \neq 0$, g is not a unit. Since M is torsion, $g \neq 0$. We now have $R/(f_1) \oplus \cdots \oplus R/(f_k) \cong R/(g)$ and both expressions are in invariant factor form. By the uniqueness of the invariant factor form, $k = 1$, $f_1 = g$, and there is a single invariant factor.