Math 200b Winter 2022 Homework 5

Due 2/25/2020 by midnight on Gradescope, or hand in in class

1. Let R be a commutative domain which is an algebra over the field F, such that $\dim_F R = n < \infty$. Prove that R is a field. (Hint: for $0 \neq a \in R$, consider the map $R \to R$ given by left multiplication by a.)

2. Let $F \subseteq K$ be a field extension with $[K : F] = 2$. If $a \in K$, by $\sqrt{a}$ we mean an element of K whose square is a.

(a) Assume that F does not have characteristic 2. Show that $K = F(\sqrt{a})$ for some $a \in F$. (Hint: think about completing the square).

(b). Show that the result of part (a) fails in general if F has characteristic 2.

3.(a). Let $F \subseteq K$ be a field extension. Suppose that $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ where K_1 and K_2 are subfields of K. The composite field K_1K_2 is defined to be the smallest subfield of K containing both K_1 and K_2. Show that if $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ then

$$K_1K_2 = \left\{ \sum_{i=1}^{d} a_i b_i \middle| a_i \in K_1, b_i \in K_2, d \geq 0 \right\}.$$

(Hint: show that the right hand side is a subring of K— then why is it forced to be a subfield?)

(b). Show that if $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ as in (b), then there is a surjective homomorphism of F-algebras $\theta : K_1 \otimes_F K_2 \to K_1K_2$ given by $\theta(a \otimes b) = ab$. Prove that θ is an isomorphism if and only if $[K_1K_2 : F] = [K_1 : F][K_2 : F]$.

(c). Show that $\mathbb{Q}(\sqrt{3}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[3]{2}) \cong \mathbb{Q}(\sqrt{3}, \sqrt[3]{2})$ as $\mathbb{Q}$-algebras.

4. Find explicitly the splitting field K of $x^6 - 4$ over $\mathbb{Q}$, and find $[K : \mathbb{Q}]$.

5. Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{C}$ and let f be the minimal polynomial of α over $\mathbb{Q}$.

(a). Compute f.

(b). Let K be a splitting field for f over $\mathbb{Q}$. Find $[K : \mathbb{Q}]$.

6. Suppose that $f \in F[x]$ has degree n and that K is a splitting field for f over F. Show that $[K : F] \leq n!$.

7. Let F be a field of characteristic p. Recall that we proved that if $f \in F[x]$ is inseparable and irreducible, then $f = \sum_{i=1}^{k} b_i x^{ip}$ for some $b_i \in F$, or in other words $f = g(x^p)$ where $g = \sum_{i=1}^{k} b_i x^i \in F[x]$.

(a). Prove that any irreducible polynomial $f \in F[x]$ is of the form $g(x^{p^k})$ for some irreducible, separable polynomial $g \in F[x]$ and some $k \geq 0$.

(b). An algebraic extension $F \subseteq K$ is called *purely inseparable* if for all $\alpha \in K - F$, the minimal polynomial of α over F is inseparable. Prove that $F \subseteq K$ is purely inseparable if and only if every $\alpha \in K$ satisfies $\alpha^{p^k} \in F$ for some $k \geq 0$.