Math 200b Winter 2022 Homework 4

Due 2/11/2020 by midnight on Gradescope, or hand in in class

1. Let V be a vector space over a field F, with $v_1, v_2 \in V$ be linearly independent over F. Show that $v_1 \otimes v_1 + v_2 \otimes v_2 \in V \otimes_F V$ is an element which is not equal to any pure tensor $u \otimes w$ with $u, w \in V$.

2. Let $F \subseteq K$ be an inclusion of fields. Then K is an F-algebra in a natural way. Since K and $F[x]$ are F-algebras, $K \otimes_F F[x]$ is also an F-algebra.

(a). Prove that $K \otimes_F F[x]$ is actually a K-algebra and that there is a K-algebra isomorphism $K \otimes F[x] \cong K[x]$.

(b). Consider the ring $R = F[x]/(g(x))$ for some $g \in F[x]$. Prove that $K \otimes_F R \cong K[x]/(g(x))$ as K-algebras.

(c). Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{C}$-algebras. (Hint: $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$.)

3. Let R be a commutative ring and let X be any multiplicative system in R. Let RX^{-1} be the localization of R along X. For any left R-module M, we can also define the localization of M along X. This is an RX^{-1}-module MX^{-1}, where

$$MX^{-1} = \{m/x \,|\, m \in M, x \in X\}/\sim$$

that is, MX^{-1} consists of equivalence classss of formal fractions m/x under the equivalence relation $\sim$, where $m/x \sim n/y$ if and only if there is $z \in X$ such that $z(ym - xn) = 0$. Here MX^{-1} is an abelian group under the operation $m/x + n/y = (ym + xn)/xy$ and a left RX^{-1}-module via $(r/x) \cdot (m/y) = (rm)/(xy)$. The proof that this is a well-defined module is very similar to the proof that RX^{-1} is a ring with well-defined operations. This fact can just be assumed here.

(a). Prove that for any left R-module M, we have $MX^{-1} \cong RX^{-1} \otimes_R M$ as left RX^{-1}-modules. This gives an alternative way of thinking of the localization of a module via the tensor product.

(b). Show that if $\phi : M \to N$ is an injective R-module homomorphism, then $1 \otimes \phi : RX^{-1} \otimes_R M \to RX^{-1} \otimes_R N$ defined by $[1 \otimes \phi](r/x \otimes m) = r/x \otimes \phi(m)$ is an injective RX^{-1}-module homomorphism. Conclude that RX^{-1} is a flat R-module. (hint: use (a) and work with the localizations of the modules rather than the tensor products).

4. Let R be an integral domain with field of fractions F. Let M be a finitely generated R-module. Show that $F \otimes_R M$ is a finite-dimensional F-vector space. We define the *rank* of M to be $\mathrm{rk}(M) = \dim_F(F \otimes_R M)$.

(a). If M is a torsion R-module, then $F \otimes_R M = 0$.

(b). Show that if $\mathrm{rk}(M) = r$, then r is equal to largest natural number n such that M contains a submodule N which is free of rank n.

(c). If $0 \to M \to N \to P \to 0$ is a short exact sequence of finitely generated R-modules, then $\mathrm{rk}(N) = \mathrm{rk}(M) + \mathrm{rk}(P)$.

5. Let M be a finitely generated R-module over a PID R. Prove that if M is flat, then M is free. Does the same conclusion necessarily hold if M is not assumed to be finitely generated?