Math 200b Winter 2022 Homework 3

Due 1/28/2020 by midnight on Gradescope

1. Let M be a finitely generated torsion module over the PID R. Suppose that M has s elementary divisors and t invariant factors. Define

$$S = \{n \in \mathbb{N} | M \text{ is isomorphic to a direct sum of } n \text{ cyclic modules}\}.$$

(a) Show that $s = \max(S)$.

(b) Show that $t = \min(S)$.

2. Let F be a field. Prove that if $n \leq 3$, two matrices $A, B \in M_n(F)$ are similar if and only if $\text{charpoly}(A) = \text{charpoly}(B)$ and $\text{minpoly}(A) = \text{minpoly}(B)$. Give an example to show this result does not hold for matrices in $M_4(F)$ in general.

3. Let F be an algebraically closed field. Consider the matrix

$$M = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in M_3(F).$$

Find the minimal and characteristic polynomials of M, and the rational and Jordan canonical forms of M. (The answers may depend on the characteristic of F.)

4. let F be a field. Find representatives of each of the similarity classes of matrices $A \in \text{GL}_2(F)$ such that A has multiplicative order exactly 4 in this group. Calculate exactly how many such similarity classes there are in each case, when

(a) $F = \mathbb{Q}$.

(b) $F = \mathbb{C}$.

(c) F is a field of characteristic 2.

5. Let F be a field. Let $A \in M_n(F)$ and let $f = \text{charpoly}(A)$ and $g = \text{minpoly}(A)$. Let A^t be the transpose of A.

(a) Prove that $\text{charpoly}(A^t) = f$ and $\text{minpoly}(A^t) = g$.

(b) Show that $(C_h)^t$ is similar to C_h for any companion matrix C_h (hint: both have a single invariant factor, equal to h).

(c) Show that A is similar to A^t.

6. Suppose that A is a complex 7×7 matrix such that $A^5 = 2A^4 + A^3$. Suppose that $\operatorname{rk} A = 5$ and $\operatorname{tr} A = 4$, where rk indicates the rank and tr indicates the trace of a matrix. Find the Jordan canonical form of A.

7. Let $J \in M_n(\mathbb{C})$ be a single Jordan block corresponding to the eigenvalue $\lambda \in \mathbb{C}$.

(a) If $\lambda \neq 0$, prove that the Jordan canonical form of J^2 is also a single Jordan block, with eigenvalue λ^2. (Hint: show that every eigenvector of J^2 is also an eigenvector of J.)

(b) If $\lambda = 0$, prove that the Jordan form of J^2 consists of two Jordan blocks, of size $n/2$ and $n/2$ if n is even and of size $(n+1)/2$ and $(n-1)/2$ if n is odd.

(c) Determine necessary and sufficient conditions for a matrix $M \in M_n(\mathbb{C})$ to have a square root, that is for there to exist $N \in M_n(\mathbb{C})$ such that $N^2 = M$.