Math 200b Winter 2022 Homework 2

Due 1/21/2020 by midnight on Gradescope

1. Let G be a finite group and F a field. Recall that for a vector space V over F, the group $\mathrm{GL}(V)$ is the group of all bijective linear transformations from V to itself, with group operation equal to composition. A *representation* of the group G (over F) is a homomorphism of groups $\phi : G \to \mathrm{GL}(V)$ for some vector space V.

Let FG be the group algebra of G over F. Show that there is a one-to-one correspondence between FG-modules and representations of G over F. (This exercise shows that the study of representation theory of a finite group really just amounts to the study of modules over group algebras).

2. Let R be an integral domain which is noetherian (that is, such that every ideal of R is finitely generated as an ideal).

(a) Show that if every finitely generated torsionfree R-module is free, then R is a PID.

(b) Show that if every torsionfree R-module is free, then R is a field. (Hint: you might want to do 3(d) first).

3. Let R be an integral domain. A left R-module M is called *divisible* if given any $0 \neq r$ and $m \in M$, there exists $n \in M$ such that $rn = m$. Informally, m can be "divided" by r. Divisible modules are "large" in the sense that they are rarely finitely generated modules as we see in this problem for PIDs.

(a) Show that if M and N are R-modules, then $M \oplus N$ is divisible if and only if both M and N are divisible.

(b) Show that if a nonzero cyclic module of the form R/I is divisible, then $I = 0$ and R is a field.

(c) Assume that R is a PID for this part. Show that if R has a nonzero module M which is finitely generated and divisible, then R is a field (use the classification theorem of modules over PIDs. The result is true without assuming that R is a PID, but the proof uses more commutative ring theory than we have seen at this point.)

(d) Give an example of a divisible R-module. By part (c) it will have to be an infinitely generated R-module unless R is a field.

4. In class we proved that a finitely generated module over a PID is the direct sum of its torsion submodule and a torsionfree (in fact, free) module. This exercise shows that the same is not true of infinitely generated modules in general.

Let $M = \prod_p \mathbb{Z}/p\mathbb{Z}$, where the product runs over all distinct (positive) prime numbers p. Consider M as an abelian group (i.e. $\mathbb{Z}$-module).

(a) Show that the torsion submodule of M is $\operatorname{Tors}(M) = \bigoplus_p \mathbb{Z}/p\mathbb{Z}$.

(b) Show that $N = M/\operatorname{Tors}(M)$ is a divisible $\mathbb{Z}$-module.

(c) Show that M has no nonzero divisible $\mathbb{Z}$-submodules.

(d) Suppose that we have an internal direct sum $M = T \oplus F$, where T is torsion and F is torsionfree. Prove that $T = \operatorname{Tors}(M)$ and $F \cong M/\operatorname{Tors}(M)$. Derive a contradiction from (b) and (c).