Math 200b Winter 2022 Homework 1

Due 1/14/2020 by midnight on Gradescope

1. Let R be an integral domain. An R-module is called *torsion* if for every $m \in M$, there is $0 \neq r \in R$ such that $rm = 0$. Let M, N and $M_1, M_2, M_3, \ldots$ be torsion R-modules. For each question, either prove or give an explicit counterexample.

(a). Must the direct sum $\bigoplus_{n \geq 1} M_n$ be a torsion R-module?

(b). Must the direct product $\prod_{n \geq 1} M_n$ be a torsion R-module?

(c). Since R is commutative, $\mathrm{Hom}_R(M, N)$ is again an R-module. Must it be a torsion R-module? Does the answer change if M is a finitely generated module? What about if N is a finitely generated module?

2. Let V be a finite dimensional vector space over a field F, with basis $v_1, \ldots, v_n$. Define a linear transformation $\phi : V \to V$ by $v_i \mapsto v_{i+1}$ if $1 \leq i \leq n-1$, and $v_n \mapsto 0$. Consider the $F[x]$-module ${}_\phi V$, that is, the uniquely determined $F[x]$-module structure on V for which $x \cdot v = \phi(v)$.

Show that V has a series of $F[x]$-submodules $V_0 = \{0\} \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ where $\dim_F V_i = i$ for all i. Prove that the V_i are the only $F[x]$-submodules of V. (A module with finitely many submodules which are totally ordered under inclusion is called *uniserial.*)

3. As remarked in the notes, if R is a ring and M and N are left R-modules, then $\mathrm{Hom}_R(M, N)$ is an abelian group but it is has no additional structure in general. In this exercise we show that $\mathrm{Hom}_R(M, N)$ does obtain a module structure if M or N has the additional structure of a bimodule.

Let R and S be rings. An abelian group M is an (R, S)-*bimodule* if it is both a left R-module and a right S-module, and these two actions are compatible in the sense that $(rm)s = r(ms)$ for all $r \in R$, $m \in M$, $s \in S$. For example, R is an (R, R)-bimodule, where R acts on both the left and right by multiplication.

(a). Suppose that M is an (R, S)-bimodule and N is a left R-module. Show that $\mathrm{Hom}_R(M, N)$ is a left S-module using the action $s \cdot \phi$, where $[s \cdot \phi](m) = \phi(ms)$ for $s \in S$, $\phi \in \mathrm{Hom}_R(M, N)$, $m \in M$.

(b). Suppose that M is a left R-module and N is an (R, T)-bimodule. Show that $\mathrm{Hom}_R(M, N)$ is a right T-module using the action $\phi \cdot t$, where $[\phi \cdot t](m) = \phi(m)t$ for $t \in T$, $\phi \in \mathrm{Hom}_R(M, N)$, $m \in M$.

(c). Suppose that M is an (R, S)-bimodule and N is an (R, T)-bimodule. By parts (a) and (b), $\mathrm{Hom}_R(M, N)$ is both a left S-module and a right T-module. Show that in fact $\mathrm{Hom}_R(M, N)$ is an (S, T)-bimodule.

4. Let R be a commutative ring. If I is an ideal of R and M is an R-module, we write $IM = \{\sum_{i=1}^n x_i m_i | x_i \in I, m_i \in M\}$, which is an R-submodule of M.

(a). Let I be an ideal of R. Show that if M is an R-module, then M/IM is an R-module which is also an R/I-module via the action $(r + I) \cdot (m + IM) = rm + IM$.

(b). Recall that two sets X and Y have the same cardinality, written $|X| = |Y|$, if there is a bijective function $f : X \to Y$. Continue to assume that R is commutative, and suppose that M is a free R-module with basis X, and N is a free R-module with basis Y. Show that $M \cong N$ as R-modules if and only if $|X| = |Y|$. (Hint: If $M \cong N$, pick any maximal ideal I of R and show that $M/IM \cong N/IN$ is an isomorphism of vector spaces over the field $F = R/I$. Assume without proof the theorem from linear algebra that any two bases of a vector space have the same cardinality.)

5. In this problem you will see that the property proved in problem $4(b)$, called *invariance of basis number*, fails for free modules over noncommutative rings in general.

Let K be a field and let V be a countable-dimensional vector space over K with basis $v_1, v_2, v_3, \dots$. Let $R = \mathrm{End}_K(V)$, the ring of all K-linear transformations of V, where the ring product is function composition as always. Let $\phi \in R$ be given by $\phi(v_i) = v_{i/2}$ for all even i, and $\phi(v_i) = 0$ for odd i. Similarly let $\psi \in R$ be given by $\psi(v_i) = v_{(i+1)/2}$ for all odd i, and $\psi(v_i) = 0$ for all even i.

Show that R is an internal direct sum $R = R\phi \oplus R\psi$. Show also that $R\phi \cong R \cong R\psi$ as left R-modules. Conclude that there is an isomorphism of left R-modules $R \cong R \oplus R$. So the free modules of rank 1 and 2 over R are isomorphic.

6. A left R-module M is called *simple* or *irreducible* if the only submodules of M are 0 and M.

(a). Show that if R is commutative, the simple R-modules are exactly the cyclic left modules of the form R/P for maximal ideals P of R.

(b). Let M be a simple module over any ring R. Show that the ring $\mathrm{End}_R(M)$ is a division ring, that is, that every nonzero element of this ring is a unit. This result is called *Schur's Lemma.*

(c). The course notes showed that for a field F, the vector space of length n column vectors $V = F^n$ is a left $R = M_n(F)$-module by left matrix multiplication, and V is a simple R-module. What is $\mathrm{End}_R(V)$ in this case?