Math 200b Winter 2021 Homework 3

Due 1/29/2020 by midnight on Gradescope

1. Let $F$ be a field. Prove that if $n \leq 3$, two matrices $A, B \in M_n(F)$ are similar if and only if $\text{charpoly}(A) = \text{charpoly}(B)$ and $\text{minpoly}(A) = \text{minpoly}(B)$. Give an example to show this result does not hold for matrices in $M_4(F)$ in general.

2. Let $F$ be an algebraically closed field. Consider the matrix
   \[
   M = \begin{pmatrix}
   0 & -1 & -1 \\
   0 & 0 & 0 \\
   -1 & 0 & 0
   \end{pmatrix} \in M_3(F).
   \]
   Find the minimal and characteristic polynomials of $M$, and the rational and Jordan canonical forms of $M$. (The answers may depend on the characteristic of $F$.)

3. Consider the three matrices
   \[
   A = \begin{pmatrix}
   1 & 1 & 1 & 1 \\
   0 & 1 & 0 & 1 \\
   0 & 0 & 1 & 1 \\
   0 & 0 & 0 & 1
   \end{pmatrix}, \quad
   B = \begin{pmatrix}
   1 & 0 & 0 & 1 \\
   0 & 1 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 1
   \end{pmatrix}, \quad
   \text{and} \quad
   C = \begin{pmatrix}
   1 & 1 & 1 & 1 \\
   0 & 1 & 0 & -1 \\
   0 & 0 & 1 & 1 \\
   0 & 0 & 0 & 1
   \end{pmatrix}.
   \]

   Are any of these matrices similar to each other over $\mathbb{C}$? Justify your answer.

4. Let $F$ be a field. Find representatives of each of the similarity classes of matrices $A \in \text{GL}_2(F)$ such that $A$ has multiplicative order exactly 4 in this group, and thus calculate exactly how many such similarity classes there are, when
   
   (a) $F = \mathbb{Q}$.
   (b) $F = \mathbb{C}$.
   (c) $F$ is a field of characteristic 2.
5. Let $F$ be a field. Let $A \in M_n(F)$ and let $f = \text{charpoly}(A)$ and $g = \text{minpoly}(A)$. Let $A^t$ be the transpose of $A$.
   (a) Prove that $\text{charpoly}(A^t) = f$ and $\text{minpoly}(A^t) = g$.
   (b) Prove that $f = g$ if and only if $A$ is similar to the companion matrix $C_f$.
   (c) Show that $(C_h)^t$ is similar to $C_h$ for any companion matrix $C_h$.
   (d) Show that $A$ is similar to $A^t$.

6. Let $J \in M_n(\mathbb{C})$ be a single Jordan block corresponding to the eigenvalue $\lambda \in \mathbb{C}$.
   (a) If $\lambda \neq 0$, prove that the Jordan canonical form of $J^2$ is also a single Jordan block, with eigenvalue $\lambda^2$.
   (b) If $\lambda = 0$, prove that the Jordan form of $J^2$ consists of two Jordan blocks, of size $n/2$ and $n/2$ if $n$ is even and of size $(n + 1)/2$ and $(n - 1)/2$ if $n$ is odd.
   (c) Determine necessary and sufficient conditions for a matrix $M \in M_n(\mathbb{C})$ to have a square root, that is for there to exist $N \in M_n(\mathbb{C})$ such that $N^2 = M$.  
