MATH 200B WINTER 2020 MIDTERM–SAMPLE SOLUTIONS

1. Recall that a module M is *indecomposable* if it cannot be written as an internal direct sum $M = N_1 \bigoplus N_2$ for nonzero submodules N_1 and N_2. Let M be a finitely generated nonzero torsion module over a PID R. Show that M is indecomposable if and only if it has a single elementary divisor.

Solution. Because M is finitely generated, the fundamental theorem of modules over a PID applies. Thus

$$M \cong R^s \oplus R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_m^{e_m})$$

where the elementary divisors of M, the prime powers $p_1^{e_1}, \ldots, p_m^{e_m}$, are uniquely determined by M up to order and replacing the p_i by associates. Since M is torsion, we know that $s = 0$ and so

$$M \cong R/(p_1^{e_1}) \oplus \cdots \oplus R/(p_m^{e_m}).$$

Remark 0.1. *It is the prime powers that are called the elementary divisors, not the primes p_i themselves.*

Now if $m \geq 2$, then above we have $M \cong N_1 \oplus N_2$ where $N_1 = R/(p_1^{e_1})$ and $N_2 = R/(p_2^{e_2}) \oplus \cdots \oplus R/(p_m^{e_m})$. Since each $R/(p_i^{e_i})$ is nonzero, N_1 and N_2 are nonzero. Thus N is decomposable (i.e. not indecomposable).

On the other hand, suppose that $m = 1$, i.e. that M has a single elementary divisor and so $M \cong R/(p^e)$. We claim that M is indecomposable.

Method 1. Note that by submodule correspondence, the submodules of $R/(p^e)$ are all of the form $I/(p^e)$ for submodules I of R (i.e. ideals) such that $(p^e) \subseteq I \subseteq R$. Since R is a PID, $I = (a)$ where $a|p^e$, so we can take $I = (p^i)$ for $0 \leq i \leq e$. Now suppose that we have an internal direct sum $M \cong N_1 \oplus N_2$ where $N_1 = (p^i)/(p^e)$ and $N_2 = (p^j)/(p^e)$. Then $N_1 \cap N_2 = (p^{\max(i,j)})/(p^e)$. In particular, if N_1 and N_2 are nonzero, then $i < e$ and $j < e$ and so $\max(i, j) < e$ and $N_1 \cap N_2 \neq 0$, contradicting that we have an internal direct sum. So M is indecomposable.

Remark 0.2. *A number of students essentially used Method 1 or something similar, but without noticing that submodules of $R/(p^e)$ have such a special form, and so had messier calculations.*

Method 2. Suppose that $M = R/(p^e)$ is decomposable, say $R/(p^e) \cong N_1 \oplus N_2$, with N_1 and N_2 nonzero. Note that there is a surjective module homomorphism $R/(p^e) \to N_1$

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given by projection onto the first summand. Since $R/(p^e)$ is finitely generated as an R-module (even cyclic), the same is true of N_1. Similarly, N_2 is finitely generated. Also, since $R/(p^e)$ is torsion, so are N_1 and N_2 since this property passes to submodules. Thus the fundamental theorem applies to N_1 and N_2 as well, and $N_1 \cong R/(q_1^{e_1}) \oplus \cdots \oplus R/(q_n^{e_n})$, $N_2 \cong R/(t_1^{f_1}) \oplus \cdots \oplus R/(t_k^{f_k})$. Then

$$M \cong R/(q_1^{e_1}) \oplus \cdots \oplus R/(q_n^{e_n}) \oplus R/(t_1^{f_1}) \oplus \cdots \oplus R/(t_k^{f_k}).$$

But this shows that $q_1^{e_1}, \dots, q_n^{e_n}, t_1^{f_1}, \dots, t_k^{f_k}$, is a set of elementary divisors for M. Since the number of elementary divisors is uniquely determined by M and $n, k \geq 1$, this is impossible. So M is indecomposable.

Remark 0.3. *A number of students successfully used Method 2, but didn't justify why N_1 and N_2 are finitely generated. It is not true in general that submodules of a finitely generated module are finitely generated. A module for which this is true is called noetherian. If you know the result that a finitely generated module over a noetherian ring is a noetherian module, then since a PID is noetherian you could conclude that N_1 and N_2 are finitely generated. But above I just relied on the result that a direct summand of a finitely generated module is finitely generated, which is true over any ring. Math 200C should cover the details about the noetherian property for modules.*

2. Let S be a commutative ring and let R be a unital subring of S. Let $M_2(S)$ be the ring of 2×2-matrices with coefficients in S. Prove that

$$M_2(R) \otimes_R S \cong M_2(S)$$

as S-algebras.

Remark 0.4. *Most students correctly handled the technical checking parts of this result, so this solution will have fewer of the checking details than was expected in the exam. If students ran into trouble on this problem it was usually in showing that ϕ defined below is an isomorphism.*

Solution. Define a map $\widehat{\phi} : M_2(R) \times S \to M_2(S)$ by $(A, s) \mapsto sA$, where sA is the scalar multiplication of the matrix A by s (which is an element of $M_2(S)$). It is straightforward to check that $\widehat{\phi}$ is R-balanced (but on the exam you should check to show me you know what balanced means). Then the universal property of the tensor product implies that there is a linear map $\phi : M_2(R) \otimes_R S \to M_2(S)$ with formula $A \otimes s \mapsto sA$.

One should check that ϕ is a map of S-algebras, i.e. that it is both a ring homomorphism and an S-module homomorphism. For example, $\phi((A \otimes s)(B \otimes t)) = \phi(AB \otimes st) = stAB$ while $\phi(A \otimes s)\phi(B \otimes t) = sAtB = stAB$ since scalar multiplication commutes with matrix multiplication for matrices over a commutative ring. This shows that ϕ is multiplicative on pure tensors, but since we know that ϕ is linear and elements of $M_2(R) \otimes_R S$ are finite sums of pure tensors, it follows that ϕ is a ring homomorphism. It is also easy to check that ϕ is an S-module map, where $M_2(R) \otimes S$ is a (left) S-module via $t \cdot (A \otimes s) = A \otimes st$. (Everyone that did this on the exam did it correctly, so I omit the rest of the proof here).

Finally one needs to prove that ϕ is an isomorphism. There are two methods, both of which involve a similar idea.

Method 1. We construct an explicit inverse to ϕ. Write a 2×2 matrix as (a_{ij}) and let e_{ij} be the matrix with a 1 in the (i, j)-spot and 0 elsewhere. Define $\psi : M_2(S) \to M_2(R) \otimes_R S$ by $\psi((a_{ij})) = \sum_{i,j} e_{i,j} \otimes a_{i,j}$. It is straightforward to see that ψ and ϕ are inverses, so ϕ is an isomorphism. (On the exam I expected more details).

Method 2. We show that ϕ is injective and surjective. If $(a_{ij}) \in M_2(S)$, then we have $\phi(\sum_{i,j} e_{i,j} \otimes a_{i,j}) = (a_{ij})$, so that ϕ is surjective. Now notice that an arbitrary pure tensor in $M_2(R) \otimes S$ has the form $(a_{ij}) \otimes s$ with $(a_{ij}) \in M_2(R)$. This can be written as $\sum_{i,j} e_{i,j} \otimes a_{i,j}s$. An arbitrary element of $M_2(R) \otimes S$ is a sum of terms like this. Using bilinearity, an arbitrary element of $M_2(R) \otimes S$ then can be put in the form $\sum_{ij} e_{i,j} \otimes t_{i,j}$ for some elements $t_{i,j} \in S$. Now if $\phi(\sum_{ij} e_{i,j} \otimes t_{i,j}) = 0$, then the matrix (t_{ij}) of $M_2(S)$ is the zero matrix, so $t_{ij} = 0$ for all i, j. Then the element $\sum_{ij} e_{i,j} \otimes t_{i,j} = 0$. So ϕ is injective as required.

3. Let F be a field. A matrix $A \in M_n(F)$ is a *projection matrix* if $A^2 = A$. Show that there are precisely $n + 1$ similarity classes of projection matrices.

Solution.

The condition on A says that A satisfies the polynomial $x^2 - x = (x)(x - 1) = 0$. So the minimal polynomial of A divides $x(x - 1)$. That means that all invariant factors divide $x(x - 1)$.

Method 1. We classify the similarity matrices in terms of Jordan forms. Since all invariant factors are products of linear terms, one may decompose them into elementary divisors which are linear. Hence the elementary divisors and the Jordan forms are well defined, even though F is not assumed to be algebraically closed. (In general, to work with elementary divisors and Jordan forms one just needs the minimal polynomial to be a product of linear terms over F).

We see then that every elementary divisor is either x or $x - 1$. There are $n + 1$ choices of such elementary divisors, where we take $0 \leq i \leq n$ $(x - 1)$'s and $n - i$ x's. The corresponding Jordan form has all blocks of size 1, and so is a diagonal matrix with i ones and $n - i$ zeroes along the main diagonal. Notice that such a matrix has rank i. Every projection matrix is similar to such a diagonal Jordan form, and each such matrix is in a unique similarity class since similar matrices have the same rank.

Method 2. We classify the similarity matrices in terms of invariant factors and rational canonical forms. We seek sequences $f_1, f_2, \ldots, f_m$ such that $f_i | f_{i+1}$ for all i, and $f_m | x(x-1)$. Every such sequence consists of j x's and $(n-j)/2$ $x(x-1)$'s, or j $(x-1)$'s and $(n-j)/2$ $x(x-1)$'s, since the product of the f_i, the characteristic polynomial, has degree n. There are $2((n+1)/2)$ such sequences when n is odd and $2(n/2)+1$ such sequences when n is even. Since the similarity classes are in one-to-one correspondence with such sequences of invariant factors, there are $n+1$ similarlity classes.

The corresponding rational canonical forms have some number of diagonal blocks of the form $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and the remaining blocks being all the same, either $\begin{bmatrix} 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \end{bmatrix}$.