Math 200b Winter 2020 Homework 7

Due 3/6/2020 in class or by 5pm in Jake Postema's mailbox.

1. Let $K = \mathbb{F}_p(x, y)$ be rational functions in two variables over the field $\mathbb{F}_p$ of p elements. Let $F = \mathbb{F}_p(x^p, y^p)$.

(a). Show that K/F is purely inseparable, and that $[K : F] = p^2$.

(b). Show that there are infinitely many intermediate subfields between F and K, and conclude that the extension K/F does not have a primitive element.

2. In this problem you show that $G = \mathrm{Aut}(\mathbb{R})$ is trivial.

(a). Show that every element of G fixes $\mathbb{Q}$ pointwise; that is, $\mathrm{Aut}(\mathbb{R}) = \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

(b). Let $\sigma \in G$. Prove that σ takes squares to squares and hence takes the set of positive numbers to itself. Using this conclude that $a < b$ implies $\sigma(a) < \sigma(b)$.

(c). Prove that σ is a continuous function.

(d). Conclude that σ is the identity.

3. Let $F \subseteq K$ be a Galois extension and let $F \subseteq E \subseteq K$ and $F \subseteq L \subseteq K$ be intermediate fields. Show that there is an isomorphism $\theta : E \to L$ which restricts to the identity on F if and only if the subgroups $\mathrm{Gal}(K/L)$ and $\mathrm{Gal}(K/E)$ of $G = \mathrm{Gal}(K/F)$ are conjugate subgroups in G.

4. Let $F \subseteq K$ be an algebraic extension, not necessarily of finite degree.

(a). Let E be the set of elements $\alpha \in K$ such that α is separable over F. Show that E is a subfield of K containing F. (Hint: Show for any $\alpha, \beta \in E$ that $F \subseteq F(\alpha, \beta)$ is a separable extension. To do this, construct a Galois extension of F containing α and β.)

(b). With the same notation as in (a), show that K/E is purely inseparable. Thus any algebraic extension decomposes into a separable extension followed by a purely inseparable extension.

(c). Suppose that $F \subseteq L \subseteq K$ where K/F is an algebraic extension. Show that K/F is separable if and only if L/F and K/L are separable.

5. Let $p_1, \ldots, p_n$ be different prime numbers and let $E = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ as a subfield of $\mathbb{R}$.

(a). Show that $E/\mathbb{Q}$ is Galois and that $\mathrm{Gal}(E/\mathbb{Q})$ is elementary Abelian of order 2^n. (Hint: Show that the fields $\mathbb{Q}(\sqrt{k})$ are all different as k runs over the $2^n - 1$ different products of distinct members of the set $\{p_1, \ldots, p_n\}$.)

(b). Show that $E = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_n}$. (Hint: determine how the 2^n elements of the Galois group G act on the elements $\sqrt{p_1}, \ldots, \sqrt{p_n}$. Then show that the orbit of α under G contains 2^n different elements.)

6. Let p be a prime. Let K be the splitting field of $f(x) = x^p - 2$ over $\mathbb{Q}$. Let ζ be any primitive pth root of 1 in $\mathbb{C}$, and let $\alpha = \sqrt[p]{2} \in \mathbb{C}$.

Show that $K = \mathbb{Q}(\zeta, \alpha)$ and that $[K : F] = p(p-1)$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$ be the Galois group. Show that G is isomorphic to the semidirect product $\mathbb{Z}_p \rtimes_\phi (\mathbb{Z}_p)^*$, where $(\mathbb{Z}_p)^*$ is the group of units of $\mathbb{Z}_p$, and $\phi : (\mathbb{Z}_p)^* \to \mathrm{Aut}(\mathbb{Z}_p)$ is the natural identification.