Math 200b Winter 2020 Homework 6

Due 2/28/2020 in class or by 5pm in Jake Postema's mailbox.

1. Show that $x^2 + y^2 - 1$ is an irreducible polynomial in the ring $\mathbb{Q}[x, y]$.

2. Let K be the splitting field of $x^6 - 4$ over $\mathbb{Q}$. Find $[K : \mathbb{Q}]$.

3. Let $\alpha = \sqrt{2 + \sqrt{2}} \in \mathbb{C}$ and let f be the minimal polynomial of α over $\mathbb{Q}$.
(a). Compute f.
(b). Let K be a splitting field for f over $\mathbb{Q}$. Find $[K : \mathbb{Q}]$.

4. An algebraic field extension $F \subseteq K$ is called *normal* if whenever $g \in F[x]$ is irreducible and $\alpha \in K$ is a root of g, then g splits completely into linear factors in $K[x]$. Let $F \subseteq K$ be a field extension such that $[K : F] < \infty$. Prove that $F \subseteq K$ is a normal extension if and only if K is the splitting field over F of some polynomial $f \in F[x]$.

Hint: Suppose that K is a splitting field over F for a polynomial $f \in F[x]$. Let $g \in F[x]$ be an irreducible polynomial with a root in K. Let $K \subseteq L$ where L is a splitting field over K for g. Show that if $\alpha_1, \alpha_2 \in L$ are any two roots of g then there is an automorphism σ of L which is the identity on F and which takes α_1 to α_2. Then notice that $\sigma(K) \subseteq K$. The other direction is easier.

5. Let $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ be fields. Suppose that $F \subseteq K_1$ and $F \subseteq K_2$ are algebraic extensions which are normal.
(a). Show that $F \subseteq K_1 \cap K_2$ is also a normal extension.
(b). Suppose that $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$. Show that $F \subseteq K_1K_2$ is a normal extension. (The composite K_1K_2 is defined on the previous homework.)

6. Let F be a field of characteristic p. Recall that we proved that if $f \in F[x]$ is inseparable and irreducible, then $f = g(x^p)$ for some $g \in F[x]$.
(a). Prove that any irreducible polynomial $f \in F[x]$ is of the form $g(x^{p^k})$ for some irreducible, separable polynomial $g \in F[x]$.
(b). An algebraic extension $F \subseteq K$ is called *purely inseparable* if for all $\alpha \in K \setminus F$, the minimal polynomial of α over F is inseparable. Prove that $F \subseteq K$ is purely inseparable if and only if every $\alpha \in K$ satisfies $\alpha^{p^k} \in F$ for some $k \geq 0$.
(c). Show that if F is a non-perfect field, for all $k \geq 1$ there is a purely inseparable extension $F \subseteq K$ with $[K : F] = p^k$.