# Math 200b Winter 2020 Homework 5

Due 2/21/2020 in class or by 5pm in Jake Postema's mailbox.

1. Let $F$ be a field of characteristic not 2. Let $F \subseteq K$ be a field extension, and let $a, b \in F$ be elements, neither of which is a square in $F$. Let $\sqrt{a}, \sqrt{b} \in K$ be roots of the polynomials $x^2 - a, x^2 - b \in F[x]$, respectively. Prove that $[F(\sqrt{a}, \sqrt{b}) : F] = 4$ if and only if $ab$ is not a square in $F$, and $[F(\sqrt{a}, \sqrt{b}) : F] = 2$ otherwise.

2. Let $R$ be a commutative domain which is an algebra over the field $F$, such that $\dim_F R = n < \infty$. Prove that $R$ is a field. (Hint: for $0 \neq a \in R$, consider the map $R \to R$ given by left multiplication by $a$.)

3(a). Let $F \subseteq K$ be a field extension. Suppose that $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ where $K_1$ and $K_2$ are subfields of $K$. Following the definition in the text, the composite field $K_1K_2$ is defined to be the smallest subfield of $K$ containing both $K_1$ and $K_2$. Show that if $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ then $K_1K_2$ can also be described as the usual notation for products of subsets of a ring suggests:

$$K_1K_2 = \left\{ \sum_{i=1}^{d} a_i b_i \middle| a_i \in K_1, b_i \in K_2, d \geq 0 \right\} \subseteq K.$$

(Hint: problem 2).

(b). Show that if $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ as in (b), then there is a surjective homomorphism of $F$-algebras $\theta : K_1 \otimes_F K_2 \to K_1K_2$ given by $\theta(a \otimes b) = ab$. Prove that $\theta$ is an isomorphism if and only if $[K_1K_2 : F] = [K_1 : F][K_2 : F]$.

(c). Show that the $\mathbb{Q}$-algebra $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$ is a field which is isomorphic to $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

4. Let $F \subseteq K$ be a field extension. Suppose that $K = F(\alpha_1, \ldots, \alpha_n)$, where $f(x) = (x - \alpha_1) \ldots (x - \alpha_n) \in F[x]$. Show that $[K : F] \leq n!$.