Math 200b Winter 2020 Homework 4

Due 2/7/2020 in J. Postema's mailbox by 5pm (no class that day)

1. Let V be a vector space over a field F and let $v_1, v_2 \in V$ be linearly independent over F. Show that $v_1 \otimes v_2 + v_2 \otimes v_1 \in V \otimes_F V$ is an element which is not equal to any pure tensor $u \otimes w$ with $u, w \in V$.

2. Let R be a commutative ring and let I be an ideal of R. Let M be a left R-module.

(a). Prove that $R/I \otimes M \cong M/IM$ as left R-modules.

(b). Suppose that I and J are ideals of R. Prove that there is an isomorphism of R-algebras

$$R/I \otimes_R R/J \cong R/(I+J)$$

(that is, an isomorphism of R-modules that is also a ring isomorphism).

3. Let $F \subseteq K$ be an inclusion of fields. Then K is an F-algebra in a natural way. Since K and $F[x]$ are F-algebras, $K \otimes_F F[x]$ is also an F-algebra.

(a). Prove that $K \otimes_F F[x]$ is actually a K-algebra and that there is a K-algebra isomorphism $K \otimes F[x] \cong K[x]$.

(b). Consider the ring $R = F[x]/(g(x))$ for some $g \in F[x]$. Prove that $K \otimes_F R \cong K[x]/(g(x))$ as K-algebras.

(c). Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{C}$-algebras. (Hint: $\mathbb{C} \cong \mathbb{R}[x]/(x^2+1)$.)

4. Recall from a previous homework that an (R,S)-bimodule M is a left R-module and right S-module structure on M such that $(rm)s = r(ms)$ for all $r \in R, m \in M, s \in S$. You proved on that homework that if M is an (R,S)-bimodule and N is an (R,T)-bimodule, then $\mathrm{Hom}_R(M,N)$ is naturally an (S,T)-bimodule. The following exercise is a similar result for the tensor product:

Suppose that M is an (S,R)-bimodule and N is an (R,T)-bimodule. Show that $M \otimes_R N$ is an (S,T)-bimodule, where $s \cdot (m \otimes n) = sm \otimes n$ and $(m \otimes n) \cdot t = m \otimes nt$.

5. Let R be an integral domain with field of fractions F. Let M be any R-module. Let $S = R \setminus \{0\}$. Define the *localization of M along the multiplicative system S* to be the set of (equivalence classes of) formal "fractions" $MS^{-1} = \{m/s \,|m \in M, s \in S\}$ under the equivalence relation where $m/s \sim n/t$ if and only if $tm = sn$. The proof that MS^{-1} is again an abelian group, where $m/s + n/t = (tm + sn)/(st)$, is essentially the same as for the field of fractions and you can assume this.

(a). Prove that MS^{-1} is a left F-module, where $r/s \cdot m/t = rm/st$ (you should check this is well-defined).

(b). Show that $MS^{-1} \cong F \otimes_R M$ as left F-modules. So this is a case where the "extension of scalars" has a more explicit description, as a localization.

(c). Show that if $\phi : M \to N$ is an injective R-module homomorphism, then $1 \otimes \phi : F \otimes_R M \to F \otimes_R N$ defined by $[1 \otimes \phi](f \otimes m) = f \otimes \phi(m)$ is an injective F-module homomorphism. Conclude that F is a flat R-module.

6. Let R be an integral domain with field of fractions F.

(a). Let M be a finitely generated R-module. Show that $F \otimes_R M$ is a finite-dimensional F-vector space and that $\dim_F(F \otimes_R M)$ is equal to $\mathrm{rk}(M)$, the *rank* of M as defined previously. This gives another way of thinking about rank which is often useful.

(b). Show that if $0 \to M \to N \to P \to 0$ is a short exact sequence of finitely generated R-modules, then $\mathrm{rk}(N) = \mathrm{rk}(M) + \mathrm{rk}(P)$.