Math 200b Winter 2020 Homework 3

Due 1/31/2020 in class or in J. Postema's mailbox by 5pm

1. Let F be a field. Prove that two matrices in $M_3(F)$ are similar if and only if they have the same characteristic and minimal polynomials. Give an example to show this result does not hold for matrices in $M_4(F)$ in general.

2. Let F be an algebraically closed field. Consider the matrix

$$M = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in M_3(F).$$

Find the minimal and characteristic polynomials of M, and the rational and Jordan canonical forms of M. (The answers may depend on the characteristic of F.)

3. Consider the three matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Are any of these matrices similar to each other over $\mathbb{C}$?

4. Let $A \in M_n(F)$ for a field F. Show that if A is a nilpotent matrix, then $A^n = 0$.

5. let F be a field. Find representatives of each of the similarity classes of matrices $M \in \mathrm{GL}_2(F)$ such that M has multiplicative order exactly 4 in this group, and thus calculate how many such similarity classes there are, when

(a) $F = \mathbb{Q}$.

(b) $F = \mathbb{C}$.

(c) F is a field of characteristic 2.

6. Let F be a field. Show that any matrix $M \in M_n(F)$ is similar to its transpose M^t. (Hint: Suppose that $M = C_f$ is the companion matrix of a polynomial $f \in F[x]$; we will see/saw in class that M has characteristic and minimal polynomials equal to f. Now what are the minimal and characteristic polynomials of M^t?)

7. Let $J \in M_n(\mathbb{C})$ be a single Jordan block corresponding to the eigenvalue $\lambda \in \mathbb{C}$.

(a). If $\lambda \neq 0$, prove that the Jordan form of J^2 is also a single Jordan block, with eigenvalue λ^2.

(b). If $\lambda = 0$, prove that the Jordan form of J^2 consists of two Jordan blocks, of size $n/2$ and $n/2$ if n is even and of size $(n+1)/2$ and $(n-1)/2$ if n is odd.

(c). Determine necessary and sufficient conditions for a matrix $M \in M_n(\mathbb{C})$ to have a square root, that is for there to exist $N \in M_n(\mathbb{C})$ such that $N^2 = M$.

8. Suppose that F is an algebraically closed field of characteristic 2. Determine necessary and sufficient conditions for a matrix $M \in M_n(F)$ to have a square root in this case.