Math 200b Winter 2020 Homework 2

Due 1/24/2020 in class or in J. Postema's mailbox by 5pm

1. Recall that for any integral domain R, if M is an R-module we defined the *rank* of M to be the maximum n such that M contains an R-linearly independent set $\{x_1, \ldots, x_n\}$.

(a). Let M be a finitely generated R-module over the integral domain R. Prove that we have $\text{rank}(M) = n$ if and only if M contains a submodule N such that $N \cong R^n$ and M/N is a torsion R-module.

(b). If M and P are finitely generated R-modules over the integral domain R, prove that $\text{rank}(M \oplus P) = \text{rank}(M) + \text{rank}(P)$.

2. Let R be a PID and let p be a prime element of R. Consider a left module M of the form $M = R/(p^{i_1}) \oplus \cdots \oplus R/(p^{i_n})$, where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_n$.

(a). For any $k \geq 0$, $p^k M = \{p^k m | m \in M\}$ is a submodule of M. Prove that the factor module $M^{(k)} = p^k M/p^{k+1}M$ is annihilated by (p) and hence is a left $R/(p) = F$-module, where F is a field. Show that $M^{(k)}$ is an F-vector space of dimension equal to the number of i_j such that $i_j > k$.

(b). Using part (a), show that if M is isomorphic to $R/(p^{j_1}) \oplus \cdots \oplus R/(p^{j_m})$ for some $1 \leq j_1 \leq j_2 \cdots \leq j_m$, then $m = n$ and $i_k = j_k$ for all k. (This gives a way to prove one part of the proof of the uniqueness of the elementary divisors of a module).

3. Let R be an integral domain which is noetherian (that is, such that every ideal of R is finitely generated).

(a). Show that R is a PID if and only if every finitely generated torsionfree R-module is free.

(b). Show that R is a field if and only if every torsionfree R-module is free.

4. Let $R = F[x]$ where F is a field. Suppose that

$$M = R/(x^2 - 1) \oplus R/(x^2 + 1) \oplus R/(x^2 - 2x + 1) \oplus R/(x^3 + 1).$$

(a). Assuming that $F = \mathbb{Q}$ is the rational numbers, find the elementary divisors and invariant factors of M.

(b). Assuming that $F = \mathbb{C}$ is the complex numbers, find the elementary divisors and invariant factors of M.