Math 200b Winter 2020 Homework 1

Due 1/17/2020 by 5pm in TA Jake Postema's mailbox

1. Let R be an integral domain, i.e. a commutative domain. If M is an R-module, then an element $m \in M$ is called *torsion* if there exists $0 \neq r \in R$ such that $rm = 0$. Let $\operatorname{Tor}(M) = \{m \in M | m \text{ is torsion}\}$. The module M is *torsion* if $M = \operatorname{Tor}(M)$ and M is *torsionfree* if $\operatorname{Tor}(M) = 0$.

(a). Show that $\operatorname{Tor}(M)$ is a submodule of M.

(b). Show that the factor module $M/\operatorname{Tor}(M)$ is torsionfree.

(c). Suppose that $\{M_i | i \in I\}$ is a collection of R-modules, where every M_i is a torsion R-module. Must the direct sum $\bigoplus_{i \in I} M_i$ be a torsion R-module? Must the product $\prod_{i \in I} M_i$ be a torsion R-module? In each case, prove or give a counterexample.

2. Let V be a finite dimensional vector space over a field F, with basis $v_1, \dots, v_n$. Define a linear transformation $\phi : V \to V$ by $v_i \mapsto v_{i+1}$ if $1 \leq i \leq n-1$, and $v_n \mapsto 0$. Consider the $F[x]$-module ${}_\phi V$, that is, the uniquely determined $F[x]$-module structure on V for which $x \cdot v = \phi(v)$.

Show that V has a series of $F[x]$-submodules $V_0 = \{0\} \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ where $\dim_F V_i = i$ for all i. Prove that the V_i are the only $F[x]$-submodules of V. (A module with finitely many submodules which are totally ordered under inclusion is called *uniserial*.)

3. Let R and S be rings. An Abelian group M is an (R, S)-*bimodule* if it is both a left R-module and a right S-module, and these two actions are compatible in the sense that $(rm)s = r(ms)$ for all $r \in R$, $m \in M$, $s \in S$. For example, R is an (R, R)-bimodule, where R acts on both the left and right by multiplication.

(a). Suppose that M is an (R, S)-bimodule and N is a left R-module. Show that $\operatorname{Hom}_R(M, N)$ is a left S-module using the action $s \cdot \phi$, where $[s \cdot \phi](m) = \phi(ms)$ for $s \in S$, $\phi \in \operatorname{Hom}_R(M, N)$, $m \in M$.

(b). Suppose that M is a left R-module and N is an (R, T)-bimodule. Show that $\operatorname{Hom}_R(M, N)$ is a right T-module using the action $\phi \cdot t$, where $[\phi \cdot t](m) = \phi(m)t$ for $t \in T$, $\phi \in \operatorname{Hom}_R(M, N)$, $m \in M$.

(c). Suppose that M is an (R, S)-bimodule and N is an (R, T)-bimodule. By parts (a) and (b), $\operatorname{Hom}_R(M, N)$ is both a left S-module and a right T-module. Show that in fact $\operatorname{Hom}_R(M, N)$ is an (S, T)-bimodule.

4. Let R be a commutative ring. Recall that if I is an ideal of R and M is an R-module, we write $IM = \{\sum_{i=1}^{n} x_i m_i | x_i \in I, m_i \in M\}$, which is an R-submodule of M.

(a). Let I be an ideal of R. Show that if M is an R-module, then M/IM is an R-module which is also an R/I-module via the action $(r + I) \cdot (m + IM) = rm + IM$.

(b). Recall that two sets X and Y have the same cardinality, written $|X| = |Y|$, if there is a bijective function $f : X \to Y$. Continue to assume that R is commutative, and suppose that M is a free R-module with basis X, and N is a free R-module with basis Y. Show that $M \cong N$ as R-modules if and only if $|X| = |Y|$. (Hint: If $M \cong N$, pick any maximal ideal I of R and show that $M/IM \cong N/IN$ is an isomorphism of vector spaces over the field $F = R/I$. Assume without proof the theorem from linear algebra that any two bases of a vector space have the same cardinality.)

5. In this problem you will see that the property proved in problem $4(b)$, called *invariance of basis number*, fails for free modules over noncommutative rings in general.

Let K be a field and let V be a countable-dimensional vector space over K with basis $v_1, v_2, v_3, \dots$. Let $R = \mathrm{End}_K(V)$, the ring of all K-linear transformations of V, where the ring product is function composition as always. Let $\phi \in R$ be given by $\phi(v_i) = v_{i/2}$ for all even i, and $\phi(v_i) = 0$ for odd i. Similarly let $\psi \in R$ be given by $\psi(v_i) = v_{(i+1)/2}$ for all odd i, and $\psi(v_i) = 0$ for all even i.

Show that R is an internal direct sum $R = R\phi \oplus R\psi$. Show also that $R\phi \cong R \cong R\psi$ as left R-modules. Conclude that there is an isomorphism of left R-modules $R \cong R \oplus R$. So the free modules of rank 1 and 2 over R are isomorphic.

6. A left R-module M is called *simple* or *irreducible* if the only submodules of M are 0 and M.

(a). Let $V = F^n$ be the module of column vectors of length n over a field F, and let V be a left module over the $n \times n$ matrix ring $M_n(F)$ by left multiplication. Prove that V is a simple left $M_n(F)$-module.

(b). Let M be a simple module over any ring R. Show that the ring $\mathrm{End}_R(M)$ is a division ring, that is, that every nonzero element of this ring is a unit. (This result is sometimes called *Schur's Lemma*. Hint: what can you say about the kernel and image of an endomorphism of M?)

(c). What is $\mathrm{End}_R(M)$ when $R = M_n(F)$ and $M = V$ as in part (a)?