MATH 200B WINTER 2020 FINAL

Please include at the beginning of your work the following statement: "I pledge to uphold academic integrity and use only those sources allowed by the exam instructions", and follow it by your signature. If you are submitting LaTeX, typing your name will count as your signature.

Instructions: Show all of your work. You may use the result of one part of a problem in the proof of a later part, even if you do not complete the proof of the earlier part. The exam is open book— you may use our course notes, and Dummit and Foote or another textbook if you have another favorite. You may not use online sources or anyone else's help. (If you only have an online copy of your textbook, it is fine to use that). You may quote basic theorems proved in the textbook or in class, unless the result you find trivializes the problem. Avoid quoting the results of homework problems.

1 (10 pts). (a). Let R be a PID. Suppose that a and b are elements of R. Show that

$$R/(a) \otimes_R R/(b) \cong R/(c)$$

as R-modules, for some element c, and give a formula for c in terms of a and b.

(b). Suppose that M is a finitely generated R-module of rank r and with elementary divisors $p, p^2, p^3, \ldots, p^n$ (each occurring once), for some $n \geq 1$. Let $N = M \otimes_R M$. What is the rank of N? How many times does p occur in the list of elementary divisors of N?

2 (10 pts). Let F be a field. How many similarity classes of matrices $A \in M_3(F)$ are there such that $A^3 = A$? (The answer may depend on properties of F).

3 (10 pts). Let ζ be a primitive 7th root of unity. Let $K = \mathbb{Q}(\zeta)$. Find all intermediate fields $\mathbb{Q} \subsetneq E \subsetneq K$ and for each one:

(a). Find $\alpha \in E$ such that $E = \mathbb{Q}(\alpha)$.

(b). Find $\text{minpoly}_E(\zeta)$.

Justify your answers.

Date: March 18, 2020.

4 (10 pts). Let F be an extension of $\mathbb{Q}$ such that $[F:\mathbb{Q}]=4$ and $F/\mathbb{Q}$ is not Galois.

(a). Let K be the Galois closure of F. Show that K is the splitting field over $\mathbb{Q}$ of some degree 4 polynomial in $\mathbb{Q}[x]$.

(b). Prove that $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to either S_4, A_4, or the dihedral group D_8 of order 8. Show that the dihedral case occurs if and only if there is an intermediate field E with $\mathbb{Q} \subsetneq E \subsetneq F$.

5 (15 pts). Suppose that $F \subseteq K$ is a Galois field extension with $\mathrm{Char}(F) = p > 0$ and $[K:F] = p$ such that $\mathrm{Gal}(K/F)$ is a cyclic group, say generated by σ.

(a). Consider σ as an F-linear transformation of K. Show that the minimal polynomial of this linear transformation is $x^p - 1 \in F[x]$.

(b). Find the Jordan form of σ.

(c). Using part (b), show that there is $\alpha \in K$ such that $\sigma(\alpha) = \alpha + 1$.

(d). Show that $F(\alpha) = K$, and $\mathrm{minpoly}_F(\alpha) = x^p - x - a$ for some $a \in F$.