

Math 200b Winter 2015 Homework 4

Due 2/27/2015 in class

1. Let F be a field of characteristic not 2. Let a and b be elements of F , neither of which is a square in F . prove that $[F(\sqrt{a}, \sqrt{b}) : F] = 4$ if and only if ab is not a square in F , and $[F(\sqrt{a}, \sqrt{b}) : F] = 2$ otherwise.

2. Let R be a commutative domain which is an algebra over the field F , such that $\dim_F R = n < \infty$. Prove that R is a field. (Hint: for $0 \neq a \in R$, consider the map $R \rightarrow R$ given by left multiplication by a .)

3(a). Let $F \subseteq K$ be a field extension. Suppose that $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ where K_1 and K_2 are subfields of K . Following the definition in the text, the composite field $K_1 K_2$ is defined to be the smallest subfield of K containing both K_1 and K_2 . Show that if $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ then $K_1 K_2$ can also be described as the usual notation for products of subsets of a ring suggests:

$$K_1 K_2 = \left\{ \sum_{i=1}^d a_i b_i \mid a_i \in K_1, b_i \in K_2, d \geq 0 \right\} \subseteq K.$$

(Hint: problem 2).

(b). Show that $[K_1 : F] < \infty$ and $[K_2 : F] < \infty$ as in (b), then there is a homomorphism of F -algebras $\theta : K_1 \otimes_F K_2 \rightarrow K_1 K_2$, and that this is an isomorphism if and only if one has $[K_1 K_2 : F] = [K_1 : F][K_2 : F]$.

(c). Show that the $\mathbb{Q}$ -algebra $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$ is a field which is isomorphic to $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

4. (a). Find explicitly the splitting field K of $x^4 + x^2 + 1$ over $\mathbb{Q}$, and find $[K : \mathbb{Q}]$.

(b). Find explicitly the splitting field K of $x^6 - 4$ over $\mathbb{Q}$, and find $[K : \mathbb{Q}]$.

5. The following definition, which we will cover in class, is standard: an algebraic field extension $F \subseteq K$ is *normal* if whenever $f \in F[x]$ is an irreducible polynomial with at least one root in K , then f splits completely into linear factors in $K[x]$. Your book defines normal in a nonstandard way (p. 537): they say that an algebraic extension $F \subseteq K$ is normal if K is “a splitting field over F of a collection of polynomials in $F[x]$ ”. Actually, they also don’t

define what it means to be a splitting field of a collection of polynomials, but it is easy to guess.

The two definitions of normal are the same. In this problem you prove this for finite degree extensions (the general case is not much harder), by proving

Theorem 0.1 *let $F \subseteq K$ be a field extension with $[K : F] < \infty$. Then the following are equivalent:*

1. K is a splitting field over F of some polynomial in $F[x]$.
2. For all irreducible polynomials $g \in F[x]$ with at least one root in K , g splits completely into linear factors in $K[x]$.

(Hint: The harder direction is (1) implies (2). Suppose that $g \in F[x]$ is an irreducible polynomial with a root in K . Let $K \subseteq L$ where L is a splitting field over K for g . Show that if $\alpha_1, \alpha_2 \in L$ are any two roots of g then there is an automorphism σ of L which fixes every element of F and which takes α_1 to α_2 . Then prove that any such automorphism σ must satisfy $\sigma(K) \subseteq K$.)

6. (a). Let $F \subseteq K_1 \subseteq K$ and $F \subseteq K_2 \subseteq K$ be fields. Suppose that $F \subseteq K_1$ and $F \subseteq K_2$ are algebraic extensions which are normal and of finite degree. Show that $F \subseteq K_1 K_2$ and $F \subseteq K_1 \cap K_2$ are also normal extensions.

(b). Let $F \subseteq K \subseteq L$ be fields, where $F \subseteq L$ is a normal algebraic extension. Show that $K \subseteq L$ is automatically normal.

(c). Let $F \subseteq K \subseteq L$ be fields, where $F \subseteq L$ is normal and $[L : F] < \infty$. Show that $F \subseteq K$ is normal if and only if every automorphism σ of L which fixes every element of F (that is, an element of $\text{Aut}(L/F)$) satisfies $\sigma(K) \subseteq K$.

7. (a). Let $\theta : F \rightarrow L$ be a homomorphism of fields. Let $\bar{L}$ be an algebraic closure of L . Prove that for any algebraic extension $F \subseteq K$, there is a homomorphism $\tilde{\theta} : K \rightarrow \bar{L}$ such that $\tilde{\theta}|_F = \theta$. (Hint: Use Zorn's lemma).

(b). Use part (a) to prove that if $\theta : F \rightarrow L$ is an isomorphism of fields, then there is an isomorphism $\tilde{\theta} : \bar{F} \rightarrow \bar{L}$ of their algebraic closures such that $\tilde{\theta}|_F = \theta$. Conclude in particular that any two algebraic closures of a given field F are isomorphic.

8. In this problem you show that $G = \text{Aut}(\mathbb{R}) = 1$.

(a). Show that every element of $\text{Aut}(\mathbb{R})$ fixes $\mathbb{Q}$ pointwise; that is $\text{Aut}(\mathbb{R}) = \text{Aut}(\mathbb{R}/\mathbb{Q})$.

(b). Let $\sigma \in G$. Prove that σ takes squares to squares and hence takes the set of positive numbers to itself. Using this conclude that $a < b$ implies $\sigma(a) < \sigma(b)$.

(c). Prove that σ is a continuous function.

(d). Prove that any continuous function $\mathbb{R} \rightarrow \mathbb{R}$ which fixes every element of $\mathbb{Q}$ is the identity, and so $\sigma = 1$.

(Remark: By contrast, it is possible to show that $\text{Aut}(\mathbb{C})$ is uncountable! This uses the theory of transcendence degree we will discuss later if we have time.)