

Math 200b Winter 2015 Homework 3

Due 2/13/2015 in class (same day as exam)

1. Let V be a two-dimensional vector space over a field F with basis v_1, v_2 . Show that $v_1 \otimes v_2 + v_2 \otimes v_1$ is an element of $V \otimes_F V$ which is not equal to any pure tensor $u \otimes w$.

2. Let R be a commutative ring and let I and J be ideals of R . Prove that there is an isomorphism of R -algebras

$$R/I \otimes_R R/J \cong R/(I + J).$$

That is, show there is an isomorphism of rings which is also a left R -module homomorphism.

3. Let $F \subseteq K$ be a field extension; in other words, F and K are fields and F is a subring of K . Then K is an F -module by left multiplication, and so K is an F -algebra.

(a). Note that since K and $F[x]$ are F -algebras, then $K \otimes_F F[x]$ is also an F -algebra. Prove that there is an isomorphism of K -algebras $K \otimes_F F[x] \cong K[x]$.

(b). Consider the ring $R = F[x]/(g(x))$ for some polynomial $g \in F[x]$. Then R is also an F -algebra. Prove that $K \otimes_F R \cong K[x]/(g(x))$ as F -algebras.

(c). Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$ -algebras. (Hint: remember that $\mathbb{C} \cong \mathbb{R}[x]/(x^2 + 1)$.)

4. Let R be an integral domain with field of fractions F . Assume without proof in this problem that F is a flat R -module. (We will prove a more general result in 200c; the basic argument is the same as the argument that $\mathbb{Q}$ is a flat $\mathbb{Z}$ -module which is given as Example 2 on page 401 of the text, depending on a long exercise from the previous section).

Let M be a finitely generated R -module. Recall from Section 12.1 that the *rank* of M is the maximum n such that M has an R -linearly independent set $m_1, \dots, m_n$. Equivalently, the rank is the largest n such that M has a submodule isomorphic to the free module $R^{\oplus n}$.

(a). Show that if M is a torsion R -module, then $F \otimes_R M = 0$.

(b). Show that $N = F \otimes_R M$ is a finite dimensional F -vector space, with $\dim_F N = \text{rank}(M)$. This gives another way of thinking about the rank. (Hint: If $P \subseteq M$ is an R -submodule with $P \cong R^{\oplus m}$, consider the short exact sequence $0 \rightarrow P \rightarrow M \rightarrow M/P \rightarrow 0$. If M/P is not torsion, show that M has rank larger than m . Now apply $F \otimes_R -$.)

(c). Show that if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is any short exact sequence of finitely generated R -modules, then $\text{rank}(M) = \text{rank}(M') + \text{rank}(M'')$.

5. Let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \quad (0.1)$$

be a short exact sequence of left R -modules. Let P be another left R -module, and consider the associated sequence

$$0 \rightarrow \text{Hom}_R(N, P) \rightarrow \text{Hom}_R(M, P) \rightarrow \text{Hom}_R(L, P) \rightarrow 0, \quad (0.2)$$

which in general is only left exact.

Prove that if (0.1) is split exact, then (0.2) is also an exact sequence, which is also split. Conversely, suppose that (0.2) is a short exact sequence, for all R -modules P . Then prove that the original sequence (0.1) must be split.

6. (a). Consider a ring R , left R -modules M and N , and the Abelian group $\text{Hom}_R(M, N)$. Suppose that M is an (R, S) -bimodule for some S . Prove that $\text{Hom}_R(M, N)$ is a left S -module where given $f \in \text{Hom}_R(M, N)$ and $s \in S$, we define sf to be the element of $\text{Hom}_R(M, N)$ given by $[sf](m) = f(ms)$.

(b). With the same setup as above, suppose that N is an (R, S) -bimodule instead. Prove that $\text{Hom}_R(M, N)$ is a right S -module where given $f \in \text{Hom}_R(M, N)$ and $s \in S$, we define fs to be the element of $\text{Hom}_R(M, N)$ given by $[fs](m) = f(m)s$.

7. Let M be a finitely generated left R -module. By 6(b), $N = \text{Hom}_R(M, R)$ is a right R -module, using that R is an (R, R) -bimodule. Prove that if M is a projective left R -module, then N is a projective right R -module.

8. Prove that the following are equivalent for a ring R : (i) every left R -module is projective; (ii) every left R -module is injective; (iii) every short exact sequence of left R -modules is split. (The rings for which these properties hold are called *semisimple* and an important theorem of Wedderburn states that these are exactly finite products of matrix rings over division rings, that is, rings of the form $M_{n_1}(D_1) \times \dots \times M_{n_m}(D_m)$ where the D_i are division rings and the $n_i \geq 1$.)

9. In this exercise you prove the basic important fact that every left R -module M is a submodule of an injective module E .

(a). Read the proof of Corollary 37 in the text that every $\mathbb{Z}$ -module is contained in an injective $\mathbb{Z}$ -module. Notice that M is naturally a $\mathbb{Z}$ -module (i.e. Abelian group), so let Q be an injective $\mathbb{Z}$ -module containing M as a $\mathbb{Z}$ -submodule.

(b). Show that there are natural inclusions of left R -modules

$$\text{Hom}_R(R, M) \subseteq \text{Hom}_{\mathbb{Z}}(R, M) \subseteq \text{Hom}_{\mathbb{Z}}(R, Q)$$

(the left R -action comes from problem 9).

(c). Show that for any left R -module P , we have

$$\text{Hom}_{\mathbb{Z}}(P, Q) \cong \text{Hom}_R(P, \text{Hom}_{\mathbb{Z}}(R, Q))$$

as Abelian groups. (Hint: one way is to use the adjointness between tensor and hom, Theorem 43 in the text).

(d). Using part (c), prove that $E = \text{Hom}_{\mathbb{Z}}(R, Q)$ is an injective left R -module. Note that $\text{Hom}_R(R, M) \cong M$ as left R -modules to complete the proof that M embeds in an injective module.