

Math 200b Winter 2015 Homework 2

Due 1/30/2015 in class

1. Let R be a PID and let p be a prime element of R . Consider a left module M of the form $M = R/(p^{i_1}) \oplus \cdots \oplus R/(p^{i_n})$, where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_n$.

(a). For any $k \geq 0$, $p^k M = \{p^k m \mid m \in M\}$ is a submodule of M . Prove that the factor module $M^{(k)} = p^k M / p^{k+1} M$ is annihilated by (p) and hence is a left $R/(p) = F$ -module, where F is a field. Show that $M^{(k)}$ is an F -vector space of dimension equal to the number of i_j such that $i_j > k$.

(b). Using part (a), show that if M is isomorphic to $R/(p^{j_1}) \oplus \cdots \oplus R/(p^{j_m})$ for some $1 \leq j_1 \leq j_2 \leq \cdots \leq j_m$, then $m = n$ and $i_k = j_k$ for all k . (This gives a way to prove one part of the proof of the uniqueness of the elementary divisors of a module).

2. Let F be a field. Prove that two matrices in $M_3(F)$ are similar if and only if they have the same characteristic and minimal polynomials. Give an example to show this result does not hold for matrices in $M_4(F)$ in general.

3. Let F be field. Consider the matrix

$$M = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in M_3(F).$$

Find the minimal and characteristic polynomials of M , and the rational and Jordan canonical forms of M . (The answers depend on the characteristic of F .)

4. Let $R = F[x]$ where F is a field. Suppose that

$$M = R/(x^2 - 1) \oplus R/(x^2 + 1) \oplus R/(x^2 - 2x + 1) \oplus R/(x^3 + 1).$$

(a). Assuming that $F = \mathbb{Q}$ is the rational numbers, find the elementary divisors and invariant factors of M .

(b). Assuming that $F = \mathbb{C}$ is the complex numbers, find the elementary divisors and invariant factors of M .

5. Recall that the trace $\text{tr}(A)$ of a matrix A in $M_n(F)$, where F is a field, is the sum of the diagonal entries; i.e. if $A = (a_{ij})$ then $\text{tr}(A) = \sum_{i=1}^n a_{ii}$. A matrix A is nilpotent if $A^i = 0$ for some $i \geq 1$, i.e. if it is nilpotent as an element of the ring $M_n(F)$.

- (a). Show that similar matrices have the same trace.
- (b). Show that if A is a nilpotent matrix then $A^n = 0$, (where n is the size of the matrix ring) and $\text{tr}(A) = 0$.
6. How many similarity classes of matrices $M \in \text{GL}_2(\mathbb{Q})$ are there such that M has multiplicative order exactly 4? How does the answer change if we replace $\mathbb{Q}$ by $\mathbb{C}$?
7. Let F be a field. Show that any matrix $M \in M_n(F)$ is similar to its transpose M^t .
8. Let $J \in M_n(\mathbb{C})$ be a single Jordan block corresponding to the eigenvalue $\lambda \in \mathbb{C}$.
- (a). If $\lambda \neq 0$, prove that the Jordan form of J^2 is also a single Jordan block of size n , with eigenvalue λ^2 .
- (b). If $\lambda = 0$, prove that the Jordan form of J^2 is two Jordan blocks, of size $n/2$ and $n/2$ if n is even and of size $(n+1)/2$ and $(n-1)/2$ if n is odd.
- (c). Determine necessary and sufficient conditions for a matrix $M \in M_n(\mathbb{C})$ to have a square root, that is for there to exist $N \in M_n(\mathbb{C})$ such that $N^2 = M$.
9. Suppose that F is a field of characteristic 2. Determine necessary and sufficient conditions for a matrix $M \in M_n(F)$ to have a square root in this case.