

Math 200b Winter 2015 Homework 1

Due 1/16/2015 by 3pm in homework box in basement

1. Let R be an integral domain. If M is an R -module, then an element $m \in M$ is called *torsion* if there exists $0 \neq r \in R$ such that $rm = 0$. Let $\text{Tor}(M) = \{m \in M \mid m \text{ is torsion}\}$. The module M is *torsion* if $M = \text{Tor}(M)$ and M is *torsionfree* if $\text{Tor}(M) = 0$.

(a). Show that $\text{Tor}(M)$ is a submodule of M .

(b). Show that the factor module $M/\text{Tor}(M)$ is torsionfree.

2. Let R be any ring and let M be a left R -module. Given $m \in M$, the *annihilator* of m is $\text{ann}(m) = \{r \in R \mid rm = 0\}$. The *annihilator* of M is

$$\text{ann}(M) = \{r \in R \mid rm = 0 \text{ for all } m \in M\}.$$

(a). Prove that $\text{ann}(m)$ is always a left ideal of R for any $m \in M$.

(b). Prove that $\text{ann}(M)$ is an ideal of R .

(c). Give an example of an integral domain R and an R -module M such that $\text{ann}(M) = 0$ yet M is a torsion module. (Hint: consider an infinite direct sum of torsion modules with varying annihilators; you can even take $R = \mathbb{Z}$.)

3. Let V be a finite dimensional vector space over a field F , with basis $v_1, \dots, v_n$. Define a linear transformation $\phi : V \rightarrow V$ by $v_i \mapsto v_{i+1}$ if $1 \leq i \leq n-1$, and $v_n \mapsto 0$. Consider the $F[x]$ -module ${}_{\phi}V$, that is, the uniquely determined $F[x]$ -module structure on V for which $x \cdot v = \phi(v)$. Find, with proof, all $F[x]$ -submodules of this module.

4. (a). Let R be a commutative ring. Show that $\text{Hom}_R(R, M) \cong M$ as left R -modules.

(b). Let R be any ring. Recall that the *opposite ring* R^{op} is formed by taking the same underlying abelian group as R but defining a new product $a \star b = ba$. Consider R as a left R -module by left multiplication. Prove that $\text{End}_R(R) = \text{Hom}_R(R, R) \cong R^{op}$ as rings.

5. Let R be a commutative ring. Recall that if I is an ideal of R and M is an R -module, we write $IM = \{\sum_{i=1}^n x_i m_i \mid x_i \in I, m_i \in M\}$ which is an R -submodule of M .

(a). Pick any maximal ideal I of R . Show that if M is an R -module, then M/IM is an R -module which is also an R/I -module via the action $(r + I) \cdot (m + IM) = rm + IM$.

(b). Recall that two sets X and Y have the same cardinality, written $|X| = |Y|$, if there is a bijective function $f : X \rightarrow Y$. Continue to assume that R is commutative, and suppose that M is a free R -module with basis X , and N is a free R -module with basis Y . Show that $M \cong N$ as R -modules if and only if $|X| = |Y|$. (Hint: If $M \cong N$, pick any maximal ideal I of R and show that $M/IM \cong N/IN$ is an isomorphism of vector spaces over the field $F = R/I$. Assume without proof the theorem from linear algebra that any two bases of a vector space have the same cardinality.)

6. In this problem you will see that the property proved in problem 5(b), called *invariance of basis number*, fails for free modules over noncommutative rings in general.

Let K be a field and let V be a countable-dimensional vector space over K with basis $v_1, v_2, v_3, \dots$. Let $R = \text{End}_K(V)$, the ring of all K -linear transformations of V , where the ring product is function composition as always. Let $\phi \in R$ be given by $\phi(v_i) = v_{i/2}$ for all even i , and $\phi(v_i) = 0$ for odd i . Similarly let $\psi \in R$ be given by $\psi(v_i) = v_{(i+1)/2}$ for all odd i , and $\psi(v_i) = 0$ for all even i .

Show that R is internal direct sum $R = R\phi \oplus R\psi$. Show also that $R\phi \cong R \cong R\psi$ as left R -modules. Conclude that there is an isomorphism of left R -modules $R \cong R \oplus R$. So the free modules of rank 1 and 2 over R are isomorphic.

7. A left R -module M is called *simple* or *irreducible* if the only submodules of M are 0 and M .

(a). Suppose that R is a commutative ring. Show that every simple R -module M is isomorphic to the left R -module R/I for some maximal ideal I of R .

(b). Let M be a simple module over any ring R . Show that the ring $\text{End}_R(M)$ is a division ring. (This result is sometimes called *Schur's Lemma*. Hint: what can you say about the kernel and image of an endomorphism of M ?).