Thm. M is f.g. module over a PID R.

Then \( r \in \mathbb{Z} \)

\[ 1 \quad M \cong R \oplus \cdots \oplus R \oplus R/(p_i^{e_i}) \oplus \cdots \oplus R/(p_m^{e_m}) \]

where \( p_i \) are primes, \( e_i \geq 1 \).

\( r = \text{rank of } M \)

\( p_i^{e_i} \) are elementary divisors.

\[ 2 \quad M' \text{ f.g. } M \cong M' \]

iff \( M \) and \( M' \) have same rank, same elementary divisors (up to order and associates).

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Torsionfree case.
Prop. Let $M$ be f.g. torsion-free over the PID $R$. Then $M$ is free of finite rank.

**Proof.** Let $M = Rm_1 + \cdots + Rm_n$.

Induction on $n$. If $n = 0$, $M = 0$ is free of rank 0.

Now assume $n \geq 1$ and the result holds for smaller $n$.

Consider $M/Rm_i$ and its torsion submodule $\text{tors}(M/Rm_i)$.

\[ = K/Rm_i, \text{ where } K = \{ m \in M \mid rm_i = 0 \} \text{ for some } r \in R \]  

Then $(M/Rm_i)/(K/Rm_i)$ is torsion-free.
$\cong M/K$ as modules.

Note $M/K$ is generated by $m_1 + K, \ldots, m_n + K$ but $m_1 + K = 0 + K$ so $m_2 + K, \ldots, m_n + K$ generate $M/K$. By induction $M/K$ is free of finite rank.

Since $\pi: M \to M/K$ is a surjection onto a free, $\pi$ is split so $M \cong (M/K) \oplus K$.

Just need $K$ free. In fact $K$ is free of rank 1.

$K \cong \{ \sum_{i=1}^{n} c_i m_i \mid c_i \in R, \sum c_i = 0 \}$

$K/\mathfrak{m}_i$, is torsion.
Note $K$ is f.g. (a summand of $M$) $K/12m_1$ is torsion, and f.g.
Then $\text{Ann}_R(K/12m_1) \neq 0$.
(If $x_i \mapsto x_i$ are generators, and $r_1x_i \mapsto r_1t_i$, $r_2x_i \mapsto r_2t_i$
kills the module)

Pick $x \in \text{Ann}_R(K/12m_1)$.
So $xK \subseteq 12m_1$.
Notice $xK \equiv K$ (M-torsionfree) $12m_1 \equiv R$
So it is enough to show a submodule of $R$ is free. Since $R$ is a PID submodules are $xR$
which are $0$ or $\equiv R$. Q.E.D.
Ex. If \( R = K[x, y] \)

\( K \) a field, and

\( I = xR + yR \), then

\( I \) is torsion-free but free.

Check \( I \) is not cyclic.

\( I \) is not a direct sum

of 2 smaller submodules:

if \( I = J \oplus L \), then

\( \forall x \in J, 0 \neq y \in L, 0 \neq x y \in J \cup L \), contradicting that \( J \cup L \) is direct.
Cor. If $M$ is a submodule of a f.g. free module $F$ over a PID, then $M$ is free.

(actually true for arbitrary free modules over a PID)

Torsion modules.

Def. Let $p$ be a prime (not a PID).

A module $M$ is
$p$-primary if
for all $m \leq M$, $p^m = 0$
for some $i \geq 1$.

If $M$ is $p$-primary
and $m \leq M$, $\text{ann}_{R}(m) = (p_i)$
for some $i$. If $M$ is f.g.
$\text{ann}_{R}(M) = (p_i)$ some $i$.

Proof. Let $M$ be f.g.,
torsion over a PID $R$.
Then for some primes $p_i$,
$M \cong M_{p_1} \oplus \cdots \oplus M_{p_k}$.
where $M_p; i$ is
$= \{ m \in M \mid (P_i)^j m = 0 \text{ for } j \geq 3 \}$
is a $P_i$-primary submodule of $M$.

Proof. $M / \text{Torsion}$ is cotorsion.

\[ a_{\text{Tor}}(M) = 0. \]

So $a_{\text{Tor}}(M) = (a)$

\[ a = P_1^{e_1} \ldots P_k^{e_k} \]
in $R$, $P_i$ one non-associate primary prime, $e_i \geq 1$.

Each $M_{p; i}$ is a $P_i$-primary submodule of $M$. 

Use the criterion for internal direct sums to check

\[ M \cong M_{p_1} \oplus \cdots \oplus M_{p_k} \]

Ex. if \( R = \mathbb{Z} \)

\[ \mathbb{Z}/n\mathbb{Z} \]

\[ \alpha = p_1, p_2, \ldots, p_k \]

we already know

\[ \mathbb{Z}/a \mathbb{Z} \cong \mathbb{Z}/(p_1, p_2, \ldots, p_k) \]
Prop. Let $M$ be a f.g. $P$-primary $R$-module. Then $M \cong R/(p_1) \oplus \cdots \oplus R/(p_k)$
where $i > 1$.

Idea.

Suppose $M = R/(p_1) \oplus \cdots \oplus R/(p_k)$

Look at

$PM$ - factors $R/(p)$ will die

$p \left( R \left( (p_i) \right) \right) = R(P + (p_i))$
this killed by $p i^{-1}$, $\equiv 0$

$R/(p_i^{-1})$
Also look at
\[ M[p] = \{ m \in M \mid pm = 0 \} \]
\[ = K(p^{-1} + (p^i)) \oplus \cdots \oplus K(p^{-1} + (p^n)) \]
and you can think of this as a vector space over \( K(p) \) which is a field.

**Proof (sketch).**

Look at \( pM \).

If \( \text{Ann}_K(M) = (p^n) \)
then \( \text{Ann}_K(pM) = (p^{n-1}) \).

Induct on \( n \).

\[ pM = K(p^i) \oplus \cdots \oplus K(p^i) \]
\[ C_i \otimes B \otimes C_i \]

where \( C_i \) is cyclic

\[ C_i = R_{gh} \otimes \text{cyclic} \]

Choose \( h \in M \) such that \( \phi_i = g_i \) and slow \( R_{gh} \otimes \text{cyclic} \) is also direct.

This gives the part of \( M \) except the needed copies of \( \mathbb{C} \) (p).

Look at \( M(p) \) to find those.