Lecture 4  1/11/21.

Internal direct sums.

→ if \( N_1, \ldots, N_m \subseteq M \) are submodules
then \( N_1 + \cdots + N_m = \sum_{i=1}^m N_i \) if \( \forall i \in N_i \).

which is an \( R \)-submodule.

The sum of the submodules.

Thus, \( M \) an \( R \)-module.

\( N_1, \ldots, N_m \) \( R \)-submodules of \( M \).
if (i) \( N_1 + \cdots + N_m = M \) and
(ii) \( N_i \cap (N_1 + \cdots + N_{i-1} + N_{i+1} + \cdots + N_m) = 0 \)
for all \( i \).

Then \( M \cong N_1 \oplus \cdots \oplus N_m \).
and we say \( M \) is an internal direct sum
of the \( N_i \).

PROOF: By the group version,
\( \phi: N_1 \oplus \cdots \oplus N_m \rightarrow M \)
\((n_1, \ldots, n_m) \mapsto n_1 + \cdots + n_m\)
is an $\mathbb{F}$ of Abelian groups.

Now notice $\phi$ is a homomorphism of $\mathbb{F}$-modules.

\[
\text{Def. A surjective module homomorphism } f : N \rightarrow M \text{ is split if there is a homomorphism } g : M \rightarrow N \text{ s.t. } fog = 1_M.
\]

\[
\text{Lemma 16 } f : N \rightarrow M \text{ is split } \iff N \cong M \oplus (\ker f).
\]

\[
\text{Proof: } \text{let } M' = g(M), \; fog = 1_M
\]

\[
\Rightarrow \; g \text{ is injective. Then } M' \cong M.
\]

\[
\text{We'll show } N \text{ is internal direct sum of } M' \text{ and } (\ker f) = K.
\]

- $M' \cap K = 0$: if $x \in M' \cap K$, $x = g(y), \; y \in M, \; f(x) = 0$.
\[ f(x) = f(g(x)) = y = 0 \]
So \( x = g(y) = g(0) = 0 \).

\[ M^1 + K = N : \]

If \( y \in N \), consider \( y - g(f(y)) \).

Then \[
f(y - g(f(y))) =
\frac{f(y) - f(g(f(y)))}{f(y) - f(y)}
= f(y) - f(y) = 0.
\]
So \( y - g(f(y)) \in K = \ker(f) \).
So \( y = g(f(y)) + (y - g(f(y))) \in M^1 + K \).
So \( N = M \oplus K \approx M \oplus K \).

Cor. If \( f: M \rightarrow F \)

is a \( \mathbb{R} \)-module surjection, and if \( F \) is free, then

\( f \) is split, i.e., \( M \cong \mathbb{R}(\ker f) \).
**Proof:** We want \( g : F \to M \)

\( s.t. \ f \circ g = 1_F \). Let \( \{ e_1, \ldots, e_n \} \) be a basis for \( F \). For each \( \alpha \), fix \( \alpha \in M \) s.t. \( f(\alpha) = e_\alpha \).

Now there is a unique \( g : F \to M \)

\( s.t. \ g(e_\alpha) = \alpha \) and so

\( f \circ g(e_\alpha) = e_\alpha \) for all \( \alpha \),

so \( f \circ g = 1_F \). Now apply the lemma to get \( M \cong F \otimes_R (\text{coker} f) \).

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Now: Let \( R \) be commutative.

Goal: understand \( \mathbb{F} \)-modules over a PID \( R \).

**Definition:** If \( M \) is an \( R \)-module, \( m \in M \)

\( \text{ann}_R(m) := \{ r \in R \mid rm = 0 \} \)

the annihilator of \( m \).

The annihilator of \( M \) is

\( \text{ann}_R(M) := \{ r \in R \mid rm = 0 \ \forall \ m \in M \} \).
\[ \bigcap_{i \in I} \text{ann}_R(m_i) \]

Not \( \text{ann}_R(m) \) and \( \text{ann}_R(M) \) are ideals of \( R \).

**Def.** Let \( R \) be an integral domain, \( M \) an \( R \)-module.

- \( m \in M \) is torsion if \( \text{ann}_R(m) \neq 0 \), i.e. \( rm = 0 \) for some \( r \neq 0 \) in \( R \).
- Otherwise \( m \) is non-torsion.

\( \text{Tors}(M) = \{ m \in M \mid m \text{ is torsion} \} \).

- \( M \) is torsion if \( M = \text{Tors}(M) \)
- \( M \) is torsion-free if \( \text{Tors}(M) = 0 \).

**Lemma.** \( R \) an integral domain, \( M \) a module.

1. \( \text{Tors}(M) \) is a submodule of \( M \).
2. \( M/\text{Tors}(M) \) is torsion-free.

**Pf.** If \( m_1, m_2 \in \text{Tors}(M) \),

\[ rm_1 = 0 \quad r_2m_2 = 0 \quad \text{for } r_1, r_2 \in R \]

then \( r_1 (m_1 - m_2) = 0 \)

and \( r_2m_2 = 0 \Rightarrow m_1 = m_2 \in \text{Tors}(M) \).

Also \( r_1(s m_1) = 0 \) for \( r_1 \in R \) and \( s m_1 \notin \text{Tors}(M) \).
2. If \( r \neq 0 \) and \( r(m + \text{Tor}(M)) = 0 \) then \( rm + \text{Tor}(M) = 0 \). So \( rm \in \text{Tor}(M) \).

Then \( s(rm) = 0 \) so \( s \neq 0 \).

Now \( sr \neq 0 \) so \( m \in \text{Tor}(M) \).

So \( m + \text{Tor}(M) = 0 \).

So \( M/\text{Tor}(M) \) is torsion-free.

**Ex.** \( \mathbb{Q} = \mathbb{Q} \).

A module \( M \) is torsion iff every element of \( M \) has finite order. \( M \) is torsion-free iff all \( m \neq 0 \) in \( M \) has infinite order. e.g. \( \mathbb{Q} \) is torsion-free.

e.g. \( \mathbb{Z}/(m) \) is torsion with annihilator = \( (m) \).
Modules over PIDs.

Theorem. Let $R$ be a PID.
Let $M$ be a f.g. $R$-module.
Then

1. $M \cong \mathbb{Z}^{r} \oplus R/(p_{1}) \oplus \cdots \oplus R/(p_{m})$

for some primes $p_{1}, \ldots, p_{m}$.

$r \geq 1$.

$r$ is the rank of $M$.

$p_{1}, \ldots, p_{m}$ are the elementary divisors.

2. Also, $r$ and the elementary divisors are unique, up to reordering.

The elementary divisors can be replaced by an associate.
Ruler.

1. M being f.g. is essential.
   - Q is a \( \mathcal{D} \)-module
     which is not a direct sum
     of cyclic \( \mathcal{D} \)-modules (so not free).
   - \( Q \) can't be an internal
direct sum \( N \oplus P \)
   since if \( 0 \neq N \subseteq Q \)
   \( 0 \neq P \subseteq Q \) then
   \( N \cap P \neq 0 \).

2. \( R \) being a PID is
   essential, since
   \( R x + R y \subseteq K[x, y] \)
is not a direct sum of cyclics.