Last line:

Theorem: \( (k:E) \leq K/F \) holds.

There is a bijection \( \mathfrak{p} \rightleftharpoons \text{Gal}(K/F) \)

Intermediate fields \( E \subseteq E' \subseteq K \) with subgroup \( H \) of \( \text{Gal}(K/F) \) such that \( \mathfrak{p} = \text{Fix}(H) \)

which reverses inclusion, i.e., if \( E_1 \subseteq E_2 \) then
\( \text{Gal}(K/E_2) \leq \text{Gal}(K/E_1) \)

if \( H_1 \subseteq H_2 \) then
\( \text{Fix}(H_2) \leq \text{Fix}(H_1) \)

\( |\text{Gal}(K/E)| = [K:E] \)
Also 
\[ E \subseteq \mathbb{C} \text{ is normal if } \text{Gal}(k(E) / k(E)) \cong \text{Gal}(k(F) / k(F)). \]

**Example**: \( K = \text{splitting field of } x^4 - 2 = 0 \) over \( \mathbb{Q} \). Let \( x = \sqrt[4]{2} \), \( i = \sqrt{-1} \).

The roots of \( f \) in \( K \) are 
\( \pm x, \pm ix, \pm x, \pm ix \).

So \( K = \mathbb{Q}(x, i) \)

\( f \) is irreducible by Eisenstein, so 
\[ [\mathbb{Q}(x) : \mathbb{Q}] = 4 = \deg f \]

and \( \mathbb{Q}(x) \subseteq K \) is a root of \( x^2 + 1 \).

So \[ [\mathbb{Q}(x, i) : \mathbb{Q}(x)] = 2 \]

\[ [\mathbb{Q}(x, i) : \mathbb{Q}] = 8. \]

\( K / \mathbb{Q} \) is a splitting field for \( f \) so it is normal. \( \text{Gal}(K / \mathbb{Q}) \) has order 8.
Now if \( \sigma \in G = \text{gal}(k(t)/k) \) the
\( \sigma(x) \) is a root of \( f \)
\( \sigma(i) \) is a root of \( x^2 + 1 \).
8 choices total, and \( \sigma \) is determined
by what it sends \( x \) and \( i \). So all
occur.

Let \( \sigma : x \mapsto i \quad \tau : \tau \mapsto \tau \)
\[ i \mapsto i \quad i \mapsto -i \]
\[ \tau \mapsto \tau \quad \tau \mapsto -\tau \]
\[ |<\sigma> <\tau>| = 8 \quad <\sigma> \cap <\tau> = 1 \]
So \( G = \{ 1, \sigma, \sigma^2, \sigma^3, \tau, \sigma \tau, \sigma^2 \tau, \sigma^3 \tau \} \)
\( \tau \sigma(x) = \tau(\sigma x) = -i \tau \)
\( \tau \sigma(i) = -i \)
\( \sigma^3 \tau(x) = \sigma^3(\tau x) = -i \tau \)
\( \sigma^3 \tau(i) = -i \)
So \( \tau \sigma = \sigma^3 \tau = \sigma^3 \tau \).
So \( G \cong D_8 \).
Let's find all subgroups of \( G = \mathbb{Z}_6 \).

\[ \langle c \rangle = \langle 0, 2, 4 \rangle \]

\[ \langle c^2 \rangle = \langle 0, 2, 4 \rangle \]

\[ \langle c^3 \rangle = \langle 0, 2, 4 \rangle \]

\[ \langle c^4 \rangle = \langle 0, 2, 4 \rangle \]

\[ \langle c^5 \rangle = \langle 0, 2, 4 \rangle \]

\[ \langle 1 \rangle \]

Apply \( \text{Fix}(-J) \)

\[ \mathbb{Q}(c) \leq \text{Fix}(\langle c \rangle) \]

\[ \mathbb{Q}(c(i)) : \mathbb{Q}(c) \]

\[ [\mathbb{Q}(c(i)) : \mathbb{Q}] = 2 \]

What about normality?

\[ \langle c \rangle, \langle c^2 \rangle, \langle c^3 \rangle, \langle c^4 \rangle, \langle c^5 \rangle \leq \langle c \rangle \]

The others are not normal.
So all fields are normal (and Galois) over $\mathbb{Q}$ except:

$\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{2}i)$, $\mathbb{Q}(\sqrt{6}+i)$

$\mathbb{Q}(\sqrt{2}(1-i))$

c.g. $\sqrt{2}$ is a root of $x^4-2$ but $x^4-2$ does not split over $\mathbb{Q}(\sqrt{2})$.

$\sqrt{2}(1+i)$ is a root of $x^4+8$ which does not split over $\mathbb{Q}(\sqrt{2}(1+i))$.

Also $<\sqrt{2}> \cong \text{Gal}(\mathbb{Q}(\sqrt{2},i)/\mathbb{Q})$

End. Theorem says

$G/<\sqrt{2}> \cong \text{Gal}(\mathbb{Q}(\sqrt{2},i)/\mathbb{Q})$.

Both are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Another example.

Ex. $K = \mathbb{Q}(\sqrt{2}+\sqrt{3})$, $d = \sqrt{2}+\sqrt{3}$

minp $\mathbb{Q}(d) = x^4 - 4x^2 + 2$.

roots are $\pm \sqrt{2}$, $\pm \sqrt{3}$. 

$p = \sqrt{2} - \sqrt{3}$.
Claim \( k = \text{splitting field of } x^4 - 4x^2 + 2 \)
since \( \alpha \beta = \sqrt{2 + \sqrt{2}} \sqrt{2 - \sqrt{2}} \\
= \sqrt{4 - 2} = \sqrt{2} \in k \)
\( (\alpha^2 - 2 = \sqrt{2} \in k) \)

\([k : \mathbb{Q}] = 4, k / \mathbb{Q} \) is bimodal.

What is \( \text{Gal}(k / \mathbb{Q}) \)? Group of size 4.

Any \( \sigma \in G = \text{Gal}(k / \mathbb{Q}) \) is determined by where it sends \( \alpha \), which is in \( \{ \pm \alpha, \pm \beta \} \).

Choose \( \sigma(\alpha) = \beta \).

Then \( \sigma(-\alpha) = -\beta \).

Since \( \alpha \beta = \sqrt{2} \)
\( \sigma(\alpha \beta) = \sigma(\alpha) \sigma(\beta) = \sigma(\sqrt{2}) \)

Since \( \alpha^2 - 2 = \sqrt{2} \), \( \sigma(\sqrt{2}) = \sigma(\alpha^2 - 2) = \sigma(\sqrt{2}) - 2 \)
\( \sigma(2 + \sqrt{2}) - 2 = \sigma(2 - \sqrt{2}) - 2 \)
\( \sigma(2 + \sqrt{2}) = \sigma(2 - \sqrt{2}) \)
\( \sigma(\sqrt{2}) = -\sqrt{2} \)

This time, \( \sigma(\beta) = -\alpha \).
\[\sigma: \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}, \quad \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_4.\]

\[1-1=0 \quad \text{and} \quad 6 \equiv 2 \mod 4.\]

\[
\begin{align*}
G &= \langle -7 \rangle \\
\langle -2 \rangle &\rightarrow \quad \text{Fix}(-) \quad \mathbb{Q} \\
\langle -2 \rangle &\rightarrow \quad \mathbb{Q}(\sqrt{2}) \\
\mathbb{Q}(\sqrt{2}) &\rightarrow \quad K
\end{align*}
\]

Next time: Cyclotomic polynomials
Galois group of splitting field of \(x^n-1\) over \(\mathbb{Q}\).