Tensor products.

Motivation. Let $F$ be a field. $V$ an $F$-space. $\dim_F V = \infty$. Define $V^* = \text{Hom}_F(V, F)$

"linear functionals on $V"$

"dual space of $V"$.

There is a natural pairing

$$\Theta : V^* \times V \rightarrow F$$

$$(\psi, v) \mapsto \psi(v)$$

but $\Theta$ is not linear.

(consider $V^* \times V$ as $V^{\times 2}$).

Instead:

$$\Theta(\psi + \phi, v) = (\psi + \phi)(v) = \psi(v) + \phi(v)$$

$$= \Theta(\psi, v) + \Theta(\phi, v)$$
\[ \Theta(\phi, v + w) = \Theta(v + w) = \Theta(v) + \Theta(w) = \Theta(\phi, v) + \Theta(\phi, w) \]

We say \( \Theta \) is bilinear, i.e., linear separately in each coordinate.

But \( \Theta : V^* \otimes V \rightarrow F \)
\[ (\epsilon, v) \mapsto \epsilon(v) \]

is not \( F \)-linear since
\[ \Theta((\phi, v) + (\psi, w)) = \Theta(\phi + \psi, v + w) = \Theta(\phi, v) + \Theta(\psi, w) + \Theta(\phi, w) + \Theta(\psi, v) \neq \Theta(\phi, v) + \Theta(\psi, w) \].

Want a way to replace \( \Theta \) by a linear map with the same information.

We will define a tensor product \( V^* \otimes V \) and a linear map
\[ \tilde{\Theta} : V^* \otimes V \rightarrow F \]
with the same information as \( \Theta \).
Def. Let $R$ be a ring.
Let $M$ be a right $R$-module.
Let $N$ be a left $R$-module.
Let $P$ be an Abelian group.
A function $\phi: M \times N \to P$ is $R$-balanced if

1. $\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n)$
2. $\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2)$
3. $\phi(m, n) = \phi(m, n)$

$\forall r \in R, \forall m \in M, \forall n \in N$.
Ex. M a left R-module. 
R right R-module by multiplication.

\[ \Phi: R \times M \rightarrow M. \]
\[ (r, m) \rightarrow r \cdot m \]

is R-balanced.

\[ \Phi(r, sm) = rs \cdot m \]
\[ \Phi(r, s \cdot m) = r \cdot s \cdot m \]

Def. M a right R-module, P left R-module. 
A tensor product of M and N over R 
is an abelian group \( T \) and a 
R-balanced map \( \Theta: M \times N \rightarrow T \)
\[ (m, n) \mapsto \Theta(m, n) \]
s.t. for any $R$-balanced map
\[
\phi : M \times N \longrightarrow R
\]
there is a unique homomorphism of abelian groups $\psi : T \longrightarrow P$ s.t.
\[
M \times N \xrightarrow{\phi} T \xrightarrow{\psi} P
\]
commutes.
\[
(\psi \circ \Theta = \phi).
\]

Note $\psi$ has all of the information in $\phi$ since
\[
\phi = \psi \circ \Theta.
\]

**Ev.** If $M$ is a left $R$-module,
\[
R \times M \longrightarrow M
\]
is a free $R$-module product of $R$ and $M$ over $R$. 

Thm. Tensor products are unique up to isomorphism.

Let $M$ be a right $R$-module, $N$ be a left $R$-module.

Let $\Theta_1 : M \times N \rightarrow T_1$, $\Theta_2 : M \times N \rightarrow T_2$ be tensor products. Then there is a unique isomorphism of abelian groups $\psi : T_1 \rightarrow T_2$ such that $\psi \circ \Theta_1 = \Theta_2$.

Pf. - same as other universal property proofs of uniqueness.

Thm. If $M$ is a right $R$-module, $N$ is a left $R$-module. Then there exists a tensor product $\Theta : M \times N \rightarrow T$.

Pf.
Consider \( S = \mathbb{M} \times \mathbb{N} \) write \((m, n) \in S\) as \( m \circ n \)

\( F \) will be a free \( \mathbb{Z} \)-module with basis \( \{ m \circ n \mid (m, n) \in S \} \).

So an element of \( F \) looks like

\[ a_1 (m_1 \circ n_1) + \cdots + a_k (m_k \circ n_k) \]

some \( a_i \in \mathbb{Z} \), \( m_i \in \mathbb{M} \), \( n_i \in \mathbb{N} \)

Let \( T = \mathbb{E} / \mathbb{I} \) where

\( \mathbb{I} \) is the subgroup generated by

all elements of the form

- \( (m_1 + m_2) \circ n - m_1 \circ n - m_2 \circ n \)
- \( m \circ (n_1 + n_2) - m \circ n_1 - m \circ n_2 \)
- \( m \circ n - m \circ n \)
A \ w, m_1, m_2 \in M, n, n_1, n_2 \in N, r \in R.

Let \( \Theta : M \times N \rightarrow T = F/I \)

\((m, n) \mapsto mn + I \)

So \( \Theta \) is \( R \)-balanced.

Also \( \Theta \) is a tensor product if \( \Phi : M \times N \rightarrow P \) is \( R \)-balanced. We need:

\( M \times N \overset{\Theta}{\rightarrow} T \overset{\psi}{\rightarrow} P \)

\( P \rightarrow P \)

\( \psi \) a homomorphism s.t.

\( \Phi \circ \Theta = \Phi. \)

There is a unique \( R \)-linear map \( \Phi : F \rightarrow P \)

\( \sum_{(m, n)} \Phi(m, n) \rightarrow \phi(m, n) \)
(F is true)

Since \( \phi \) is \( R \)-balanced, check \( I \subseteq \ker \phi \).

(all of the generators of \( I \)

are in \( \ker \phi \), so \( I \subseteq \ker \phi \))

so \( \phi \) induces \( \psi: \mathbb{F} / I \to P \).

\[ \text{and} \quad \psi \circ \Theta = \phi. \]

Also \( \psi \) is unique (check).

\[ \text{Rule.} \]

Give a tensor product \( \Theta: M \times N \to T \). \]
we write $T$ as $M \otimes N$

we call $M \otimes N$ the tensor product and don't write $\otimes$ in the notation.

But we write $\Theta(m, n)$ as $m \otimes n$.

Caution: an arbitrary element of $T$

is an element of the form

$$
\sum_{i=1}^{d} a_i (m_i \otimes n_i) \quad a_i \in \mathbb{R}
$$

$\quad m_i \in M, \quad n_i \in N$

by the proof.

$$
\frac{d}{dx} \left( \sum_{i=1}^{d} (m_i \otimes n_i) \right) \quad \left( \text{linearity in the first coordinate} \right)
$$

$$
= \sum_{i=1}^{d} \frac{d}{dx} (m_i \otimes n_i) \quad \text{where } m_i \otimes n_i
$$

(*) an element of $M \otimes N$ looks like

$$
\sum_{i=1}^{d} (m_i \otimes n_i) \quad \text{where } m_i \otimes n_i
$$

are pure tensors.