Math 200a (Fall 2016) - Homework 5

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Posted October 22–Due **Wed. November 2** at 3pm

1 Reading

Finish reading chapter 5, and then read Chapter 6.

2 Exercises to submit on Wed. November 2

Exercise 1. While a semidirect product $G = H \rtimes_\psi K$ depends in general on the choice of homomorphism $\psi : K \to \mathrm{Aut}(H)$, sometimes different choices of ψ lead to isomorphic semidirect products. This problem explores some cases where this happens.

(a). Suppose that $\theta \in \mathrm{Aut}(H)$ and let $\phi_\theta : \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ be the inner automorphism of $\mathrm{Aut}(H)$ given by $\rho \mapsto \theta\rho\theta^{-1}$. Let $\psi_2 = \phi_\theta \circ \psi : K \to \mathrm{Aut}(H)$. Prove that $H \rtimes_\psi K$ and $H \rtimes_{\psi_2} K$ are isomorphic groups. (Hint: Try the map $H \rtimes_\psi K \to H \rtimes_{\psi_2} K$ given by $(h, k) \mapsto (\theta(h), k)$.)

(b) Suppose that $\rho : K \to K$ is an automorphism of K and define $\psi_2 = \psi \circ \rho : K \to \mathrm{Aut}(H)$. Prove that $H \rtimes_\psi K$ and $H \rtimes_{\psi_2} K$ are isomorphic groups.

Exercise 2. Suppose that p and q are primes with $p < q$ where p divides $q - 1$. Show that there are precisely two groups of order pq up to isomorphism. Find a presentation for the non-Abelian group you find, and justify that your presentation is correct.

Exercise 3. Classify groups G of order 20 up to isomorphism (there are 5 such groups). Make sure you justify why the 5 different groups you find really are non-isomorphic.

Exercise 4. Classify groups of order 75 up to isomorphism. (Hint: Find the order of the automorphism group $\mathrm{Aut}(\mathbb{Z}_5 \times \mathbb{Z}_5)$ and show that all subgroups of order 3 in this group are conjugate. You don't need to find any of the elements of order 3 explicitly to do this.)

Exercise 5. (a). Suppose that you can show that all groups G with $|G| < 60$ are not simple. Prove that this implies that all groups G with $|G| < 60$ are solvable.

(b). It is actually true that all groups G with $|G| < 60$ are not simple, and thus all groups of order less than 60 are solvable. I encourage you to work out a full proof of this, as it is a good review of the techniques we have developed. However, just write out a proof that groups of order $24, 36$, and 48 are not simple. (Consider the kernel of a conjugation action on Sylow p-subgroups.)

Exercise 6. A subgroup K of G is *maximal* if there does not exist a subgroup H of G with $K \subsetneq H \subsetneq G$. Suppose that G is finite and has the property that every maximal subgroup of G has prime index. Prove that G is solvable, in the following steps.

(a). Prove that if P is a Sylow p-subgroup of G and $N_G(P) \leq H \leq G$ for some subgroup H, then $|G : H| \equiv 1 (\bmod p)$. (This part is true in any finite group.) (Hint: P is a Sylow p-subgroup of H and $N_G(P) = N_H(P)$.)

(b). Taking p to be the largest prime dividing the order of the group G, show that G has a normal Sylow p-subgroup.

(c). Conclude the proof by induction on the order.

Exercise 7. Let G be a finite group. A group G is called *characteristically simple* if it has no characteristic subgroups other than $\{e\}$ and G. A normal subgroup $H \trianglelefteq G$ is *minimal normal* if there is no normal subgroup $K \trianglelefteq G$ with $\{e\} \subsetneq K \subsetneq H$.

(a). Prove that if G is characteristically simple, then G is isomorphic to $H \times H \times \cdots \times H$ for some simple group H.

(Hint: Let N be a minimal normal subgroup of G. Consider the collection of subgroups of G which are internal direct products of the form $N_1 \times N_2 \cdots \times N_k$ where each N_i is minimal normal in G and each N_i is isomorphic to N. Let M be a maximal element of this collection. Show that M is characteristic in G, so $M = G$. Then show that N is simple.)

(b). Does the converse to part (a) hold?

(c). Show that if N is any minimal normal subgroup of any finite group G, then N is characteristically simple, so N is a direct product of isomorphic simple groups by part (a).