Math 200a (Fall 2016) - Homework 4

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Posted October 14–Due October 21 at 3pm

1 Reading

Read Sections 4.4-4.6, and begin to read Section 5.1-5.4.

2 Exercises to submit on Friday Oct. 14

Exercise 1. We say that a subgroup $H \leq G$ is *characteristic* in G, and write $H \operatorname{char} G$, if for all automorphisms ϕ of G, $\phi(H) = H$.

(a). Show that if $H \operatorname{char} G$, then $H \trianglelefteq G$.

(b). Let $H < K < G$, where $H \operatorname{char} K$ and $K \trianglelefteq G$. Show that $H \trianglelefteq G$.

(c). Show that if K is a cyclic subgroup of G and $K \trianglelefteq G$, then every subgroup $H \leq K$ satisfies $H \trianglelefteq G$.

(d). Show by example that $H \trianglelefteq K$ and $K \trianglelefteq G$ do not necessarily imply $H \trianglelefteq G$.

Exercise 2. Let G be a finite group with subgroups $P \leq H \leq K \leq G$, where P is a Sylow p-subgroup of G.

(a). Prove that if $P \trianglelefteq H$ and $H \trianglelefteq K$, then $P \trianglelefteq K$.

(b). Prove that $N_G(N_G(P)) = N_G(P)$.

Exercise 3. Let G be a (possibly infinite) group. let $A = \operatorname{Aut}(G)$ be the automorphism group of G, and let $I = \operatorname{Inn}(G) = \{\phi_g | g \in G\}$ be the subgroup of inner automorphisms, where $\phi_g(x) = gxg^{-1}$.

(a). Show that an element $\sigma \in A$ commutes with every element of I if and only if $g^{-1}\sigma(g) \in Z(G)$ for all $g \in G$. In particular, if G is *centerless* (that is, $Z(G) = \{e\}$), then σ is the identity function 1.

(b). Suppose now that G is a simple group. Let $\sigma \in \operatorname{Aut}(A)$ be an automorphism of the automorphism group $A = \operatorname{Aut}(G)$. Prove that $\sigma(I) = I$. Hence $I \operatorname{char} A$.

(Hint. If G is Abelian, the result is easy, so assume G is simple non-Abelian. Then $I \cong G$ and hence I is a simple group. I is normal in A, so $\sigma(I)$ is normal in A also and $I \cap \sigma(I) \trianglelefteq I$. If $I \cap \sigma(I) = \{1\}$, show that every element of $\sigma(I)$ commutes with every element of I and part (a) gives a contradiction. So...)

Exercise 4. Let P be a Sylow p-subgroup of the finite group G. Let $H \subseteq G$ be a subgroup of G.

(a). Show that there exists $g \in G$ such that $gPg^{-1} \cap H$ is a Sylow p-subgroup of H.

(b). Suppose that $H \trianglelefteq G$. Prove that $P \cap H$ is a Sylow p-subgroup of H.

(c). Suppose that $P \trianglelefteq G$. Prove that $P \cap H$ is a Sylow p-subgroup of H, and is the unique Sylow p-subgroup of H.

Exercise 5. Let $|G| = pqr$ for some distinct primes p, q, r with $p < q < r$.

Prove that G has at least one normal Sylow subgroup.

Exercise 6. Let $|G| = 595 = (5)(7)(17)$. Show that all Sylow subgroups of G are normal.

Exercise 7. Let $|G| = p(p+1)$ where p is prime. Show that G has either a normal subgroup of order p or a normal subgroup of order $p+1$. (Hint: If $n_p > 1$, choose $x \in G$ of order not equal to 1 or p. Study the conjugacy class of x and $|C_G(x)|$.)