Math 200a (Fall 2016) - Homework 2

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Posted September 30th–Due October 7th at 3pm

1 Reading

Read Section 6.3 on free groups and presentations, then begin to read Sections 4.1-4.3 of the text on group actions.

2 Exercises to submit on Friday Oct. 7

Exercise 1. Show that a group G is abelian if and only if it satisfies the property: in any triplet of elements of G, two of the elements commute.

Exercise 2. Let H_1, H_2, H_3 be subgroups of a group G. Show that

(i) if $G = H_1 \cup H_2$, then $G = H_1$ or $G = H_2$;

(ii) if G is finite and $G = H_1 \cup H_2 \cup H_3$, then either $G = H_i$ for some i, or $|G : H_i| = 2$ for all i.

Find the smallest group for which the second possibility happens in assertion (ii).

Exercise 3. Let H be a subgroup of a group G. Show that there exists a normal subgroup $\operatorname{clos}(H)$, called the *normal closure of H in G*, that enjoys the following equivalent properties:

(i) $\operatorname{clos}(H)$ is the smallest normal subgroup (of G) containing H

(ii) $\operatorname{clos}(H)$ is the subgroup generated by all the conjugates of H

(iii) $\operatorname{clos}(H) = \bigcap_{H \le N \trianglelefteq G} N$

(iv) Any morphism from G that kills H factors uniquely through $G/\operatorname{clos}(H)$.

Exercise 4. Let H, K be subgroups of a group G such that $H, K \trianglelefteq HK \le G$. Show that there exists a canonical isomorphism

$$HK/(H \cap K) \cong (HK/H) \times (HK/K).$$

Exercise 5. A subgroup H of a finite group G satisfying $\gcd(|H|, |G : H|) = 1$ is called a *Hall subgroup* of G. Let H be a Hall subgroup of G and let $N \trianglelefteq G$. Show that $H \cap N$ is a Hall subgroup of N, and that HN/N is a Hall subgroup of G/N.

Exercise 6. Recall that a homomorphism $\phi : G \to G$ from a group G to itself is called an *endomorphism*, and that an isomorphism $\phi : G \to G$ is called an *automorphism*. The set of all automorphisms $\phi : G \to G$ is written $\mathrm{Aut}(G)$ and it is easy to check that $\mathrm{Aut}(G)$ is a group under the operation of composition (you should convince yourself of this, but don't write it up).

Let G be a cyclic group of order n and for each $a \in \mathbb{Z}$, set

$$\sigma_a : G \to G : x \mapsto x^a$$

Prove that:

(i) each σ_a is an endomorphism of G, and $\sigma_a \in \mathrm{Aut}\, G$ if and only if $\gcd(a, n) = 1$;

(ii) $\sigma_a = \sigma_b$ if and only if $a = b \mod n$;

(iii) every automorphism of G is equal to σ_a for some $a \in \mathbb{Z}$;

(iv) $\sigma_a \circ \sigma_b = \sigma_{ab}$, and therefore the map $(\mathbb{Z}/n\mathbb{Z})^\times \to \mathrm{Aut}\, G : a \mapsto \sigma_a$ is an isomorphism of groups. In particular, $\mathrm{Aut}\, G$ is an abelian group of order $\varphi(n)$. (Recall that $(\mathbb{Z}/n\mathbb{Z})^\times$ denotes the group of units of the ring $\mathbb{Z}/n\mathbb{Z}$.)

Exercise 7. Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by the formulas $f(x) = -x$ and $g(x) = x+1$. Let $G = \langle f, g\rangle$ be the subgroup of $\mathrm{Sym}(\mathbb{R})$ (the group of all permutations of the set $\mathbb{R}$) generated by f and g. Prove carefully that

$$G \cong \langle a, b \,|\, b^2 = e, ba = a^{-1}b\rangle$$

(G is called the *infinite dihedral group*, D_∞.)

Exercise 8. Prove that the following is a presentation of the quaternion group:

$$Q_8 = \langle a, b \mid a^2 = b^2, a^{-1}ba = b^{-1}\rangle.$$

Exercise 9. Consider the group G given by the presentation

$$G = \langle x, y \mid xy^2 = y^3x,\ yx^2 = x^3y\rangle.$$

Show that G is trivial.
(This exercise illustrates how difficult it is to tell the shape of a group just from a presentation. To this purpose, it is recommended to first try the exercise without the following hint. Establish that $x^2y^8x^{-2} = y^{18}$ and $x^3y^8x^{-3} = y^{27}$ using the first relation. Using the second relation, deduce that $y^9 = e$. Conclude.)